Coverings, Étale Coverings, Hypercoverings, and Homotopy Coherence

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Coverings, Étale Coverings, Hypercoverings, and Homotopy Coherence

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by

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Dedicated

to

my father and mother: Abdul Kadir bin Alimuddin and Salamah binti Sirun,

my wife: Norhidayah binti Mohd. Mukhtar,

and my children: Suraya and Burhanuddin

Declaration

This work has not previously been accepted for any degree and is not being concurrently submitted in candidature for any degree.

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Summary

This thesis reproves the Classical Lemma of the Čech complex functor and later proves the lift of it to the homotopy coherent Čech complex functor.

By considering the transition from open coverings to étale coverings of a space, we proves the existence of the étale Čech complex functor and its associated lift, the homotopy coherent étale Čech complex functor. Particularly, open coverings as "inclusion" étale coverings will give the "inclusion" étale Čech complex functor and its associated lift, the homotopy coherent "inclusion" étale Čech complex functor.

And lastly, the transition to hypercoverings of a space translate those last two previous results on homotopy coherence to give the homotopy coherent canonical hypercovering functor and the homotopy coherent "inclusion" canonical hypercovering functor. By passing to the more general idea of hypercoverings of a topos, we conjectured the existence of the homotopy coherent hypercovering functor and homotopy coherent Grothendieck-Verdier hypercovering functor.

Contents

Chapter 1

Introduction

The problem of homotopy coherence has occurred in two contexts: explicitly in Strong Shape Theory, cf. Edwards and Hastings [26], and implicitly in the simplicial localization, cf. Dwyer and Kan [25], and Cordier [12]. The first context enables one to approach it from studying the generalization of homotopy limits and colimits, and its relations to homotopy coherence, cf. Bourn [5], Bourn and Cordier [6], Cordier [12], and Cordier and Porter [16].

The second approach, used for the development of this thesis, derived its studies from the notion of homotopy **A**-diagram, defined as a $T\mathbf{A}$-diagram D, such that

$D_{A,B}(f_n, t_n, ..., f_1, t_1, f_0; x) =$

$$\begin{cases} D_{A,B}(f_n, t_n, ..., f_2, t_2, f_1; x) & \text{if } f_0 = id \\ D_{A,B}(f_n, t_n, ..., f_{i+1}, t_i t_{i+1}, f_{i-1}, ..., f_1, t_1, f_0; x) & \text{if } f_i = id,\ 0 < i < n \\ D_{A,B}(f_{n-1}, t_{n-1}, ..., f_1, t_1, f_0; x) & \text{if } f_n = id \\ D_{A,B}(f_n, t_n, ..., t_{i+1}, f_i f_{i-1}, t_{i-1}, ..., f_1, t_1, f_0; x) & \text{if } t_i = 1 \end{cases}$$

with $x \in D_0 A$ and $(f_n, t_n, ..., f_1, t_1, f_0) \in T\mathbf{A}(A, B)$, where **A** is a **Top**-category, cf. Boardman and Vogt [4], and Vogt [55].

Furthermore, using an $\mathcal{S}$-category $\mathbb{S}(\mathbf{A})$ on **A**, developed by Dwyer and Kan [25] and Cordier [12], which hereafter will be called Dwyer-Kan-Cordier (DKC) $\mathcal{S}$-category, and the fact that **Top** is a (locally Kan) $\mathcal{S}$-category, denoted by $\mathbf{Top}_{\mathcal{S}}$, Cordier then showed that, up to the replacement of the monoid structure on the interval by max, a homotopy $\mathbf{A}$-diagram is equivalent to a homotopy coherent diagram of type $\mathbf{A}$ in $\mathbf{Top}_{\mathcal{S}}$, defined as a $\mathbf{S}$-functor

$$F : \mathbb{S}(\mathbf{A}) \longrightarrow \mathbf{Top}_{\mathcal{S}}.$$

This enabled him to introduce the more generalized homotopy coherent diagram, i.e. of type $\mathbf{A}$ in an arbitrary (locally Kan) $\mathcal{S}$-category $\mathbf{B}_{\mathcal{S}}$, and gives the result as an $\mathcal{S}$-functor

$$F : \mathbb{S}(\mathbf{A}) \longrightarrow \mathbf{B}_{\mathcal{S}}.$$

This interesting generalization gives us a guide to consider a homotopy coherent diagram of type $\mathbf{A}$ in other $\mathcal{S}$-categories. For example, Tonks [54] uses the homotopy coherent diagram of type $\mathbf{A}$ in $\mathcal{S}_{\mathcal{S}}$, and introduced the notion of a homotopy coherent diagram of type $\mathbf{A}$ in $\mathbf{Crs}_{\mathcal{S}}$. Cordier and Porter [15], and Brown, Golasinski, Porter and Tonks [10], [11] examine the naturally occuring problems of homotopy coherent diagram in equivariant homotopy theory.

For background, Chapter 2 gives the necessary material needed as background for this thesis. They are: the theory of simplicial objects and its application on simplicial sets, the theory of comonad resolutions, the theory of $\mathbf{V}$-categories with special attention to $\mathbf{Top}$-categories and $\mathcal{S}$-categories, and coskeleta theory.

Chapter 3 gives the detailed exposition of the above mentioned developments of homotopy coherent diagrams. They are: $\mathbf{A}$-diagrams, $\mathbf{Top}$-enriched homotopy $\mathbf{A}$-diagrams with its associated category $\mathbf{Coh(A,Top)}$, $\mathcal{S}$-enriched homotopy coherent diagrams with its associated category $\mathbf{Coh(A,Top_{\mathcal{S}})}$, and the generalized $\mathcal{S}$-enriched homotopy coherent diagrams with its associated category $\mathbf{Coh(A,B_{\mathcal{S}})}$.

Our intention in Chapter 4 is to study a specific type of naturally occuring homotopy coherent functor, called the homotopy coherent Čech complex functor

$$C(X;-) : \mathbb{S}(\mathbf{Cov_{\leq}(X)}) \longrightarrow \mathcal{S}_{\mathcal{S}}$$

which coming from the lifting of the classical Čech complex functor

$$C(X;-) : \mathbf{Cov_{\leq}(X)} \longrightarrow \mathbf{Ho}(\mathcal{S})$$

that together make the following diagram

$$\begin{array}{ccc} \mathbb{S}(\mathbf{Cov_{\leq}(X)}) & \xrightarrow{C(X;-)} & \mathcal{S}_{\mathcal{S}} \\ \Big\downarrow{\scriptstyle aug} & & \Big\downarrow{\scriptstyle \pi_0} \\ \mathbf{Cov_{\leq}(X)} & \xrightarrow[C(X;-)]{} & \mathbf{Ho}(\mathcal{S}) \end{array}$$

commute.

Chapter 5 then will analyse the transition from open coverings to étale coverings. It resulted in giving us a lifting of the étale Čech complex functor

$$\mathcal{E}(X;-) : \mathbf{\acute{E}tCov_{\leq}(X)} \longrightarrow \mathbf{Ho(SimpShv(X))}.$$

to the homotopy coherent étale Čech complex functor

$$\mathcal{E}(X;-) : \mathbb{S}(\mathbf{\acute{E}tCov_{\leq}(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}}$$

such that the following diagram

$$\begin{array}{ccc} \mathbb{S}(\mathbf{\acute{E}tCov_{\leq}(X)}) & \xrightarrow{\mathcal{E}(X;-)} & \mathbf{SimpShv(X)}_{\mathcal{S}} \\ \Big\downarrow{\scriptstyle aug} & & \Big\downarrow{\scriptstyle \pi_0} \\ \mathbf{\acute{E}tCov_{\leq}(X)} & \xrightarrow[\mathcal{E}(X;-)]{} & \mathbf{Ho(SimpShv(X))}. \end{array}$$

commute.

Particularly, taking the usual open coverings as "inclusion" étale coverings will lift the "inclusion" étale Čech complex functor

$$Inc\mathcal{E}(X;-) : \mathbf{Inc\acute{E}tCov_{\leq}(X)} \longrightarrow \mathbf{Ho(SimpShv(X))}$$

to the homotopy coherent "inclusion" étale Čech complex functor

$$Inc\mathcal{E}(X;-) : \mathbb{S}(\mathbf{Inc\acute{E}tCov_{\leq}(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}}.$$

The discussions later, in Chapter 6, will first start by the process of identifying an étale Čech complex $\mathcal{E}(X; E_X)$ and an "inclusion" étale Čech complex $Inc\mathcal{E}(X;\mathcal{U}_X)$ as a canonical hypercovering and an "inclusion" canonical hypercovering of X. We conjectured that this transition to hypercoverings will lift the functor

$$\mathbf{Ho(CanHCov(X))} \longrightarrow \mathbf{Ho(SimpShv(X))}$$

to the homotopy coherent canonical hypercovering functor

$$\mathbb{S}(\mathbf{CanHCov(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}},$$

and particularly, the functor

$$\mathbf{Ho(IncCanHCov(X))} \longrightarrow \mathbf{Ho(SimpShv(X))},$$

to the "inclusion" homotopy coherent canonical hypercovering functor

$$\mathbb{S}(\mathbf{IncCanHCov(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}}.$$

Related to the more general ideas of hypercoverings of a topos, we also conjectured the existence of the homotopy coherent hypercovering functor

$$\mathbb{S}(\mathbf{HCov(Shv(X))}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}}$$

and the homotopy coherent Grothendieck-Verdier hypercovering functor

$$\mathbb{S}(\mathbf{HCov(Shv(C)))}) \longrightarrow \mathbf{SimpShv(C)}_{\mathcal{S}}.$$

Chapter 2

Simplicial Theory

2.1 Introduction

To make more precise our intention to study in Chapter 4 a specific type of naturally occuring homotopy coherent diagram of type **Cov(X)** in $\mathcal{S}_{\mathcal{S}}$, that is, the homotopy coherent Čech complex functor

$$\mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X})) \longrightarrow \mathcal{S}_{\mathcal{S}},$$

we examine below the necessary technicalities. We discuss the theory of simplicial objects in Section 2.2, the theory of simplicial resolutions in Section 2.3, and the theory of **V**-categories in Section 2.4. Further, detailed explanations on coskeleta theory are given in Section 2.5 as a preliminary materials for the discussion of hypercoverings of the topos in Chapters 6.

2.2 Simplicial Objects

Let Δ be the category whose objects are the non-empty finite totally ordered sets $[n] = \{0 \leq 1 \leq ... \leq n\}$, where n is a non-negative integer, and in which the morphisms are maps $\mu : [m] \longrightarrow [n]$ such that $i \leq j$ implies $\mu(i) \leq \mu(j)$. For each n and $i \in [n]$, denote

$$\delta_n^i : [n-1] \longrightarrow [n]$$

$$j \mapsto \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$$

the increasing injection which leaves out i, and

$$\sigma_n^i : [n+1] \longrightarrow [n]$$
$$j \mapsto \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}$$

the increasing surjection which repeats i. These two classes of increasing maps together generate Δ.

2.2.1 Category of Simplicial Objects

Further results are taken from May [39], Bousfield and Kan [7] and Curtis [19]. To avoid difficulties in writing, all of the diagrams represented simplicial objects will have face operators d_i only, neglecting the degeneracy operators s_i.

Definition 2.2.1 Let **C** be a small category. A *simplicial object* X in **C** is defined to be a functor $X : \Delta^{op} \longrightarrow \mathbf{C}$, and is thus given by a diagram

$$\cdots \underset{d_0}{\overset{d_3}{\rightrightarrows\!\!\rightrightarrows}} X_2 \underset{d_0}{\overset{d_2}{\rightrightarrows\!\!\rightarrow}} X_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} X_0$$

which can be specified by

$$d_i = X(\delta_n^i) : X_n \longrightarrow X_{n-1}$$
$$s_i = X(\sigma_n^i) : X_n \longrightarrow X_{n+1}$$

satisfying the simplicial identities

$$d_i d_j = d_{j-1} d_i \quad \textit{for } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ Id & i = j, j+1 \\ s_j d_{i-1} & i > j+1 \end{cases}$$

$$s_i s_j = s_j s_{i-1}, \quad \textit{for } i > j.$$

For a given pair of simplicial objects, a simplicial map between them satisfies some simplicial properties.

Definition 2.2.2 Suppose X and Y are simplicial objects in $\mathbf{C}$. A *simplicial map* $f : X \longrightarrow Y$ is a natural transformation specified by morphisms

$$f_n : X_n \longrightarrow Y_n$$

such that

$$d_i f_n = f_{n-1} d_i$$
$$s_i f_{n-1} = f_n s_i,$$

for all $0 \leq i \leq n$ and for all $n > 0$.

The category of simplicial objects in $\mathbf{C}$ is denoted by **Simp(C)**. A homotopy between a pair of simplicial maps of simplicial objects is defined in the following way.

Definition 2.2.3 Suppose $f, g : X \longrightarrow Y$ are simplicial maps of simplicial objects. Then f *is homotopic to* g, written $f \simeq g$, if there is a system of arrows $\{h_i\}_{0 \leq i \leq n} : X_n \longrightarrow Y_{n+1}$ which satisfies:

$$\begin{aligned} d_0 h_0 &= f_n \\ d_{n+1} h_n &= g_n \end{aligned}$$

$$\begin{aligned} d_i h_j &= h_{j-1} d_i \quad i < j \\ d_{j+1} h_{j+1} &= d_{j+1} h_j \\ d_i h_j &= h_j d_{i-1} \quad i > j+1 \end{aligned}$$

$$\begin{aligned} s_i h_j &= h_{j+1} s_i \quad i \leq j \\ s_i h_j &= h_j s_{i-1} \quad i > j. \end{aligned}$$

2.2.2 Category of Simplicial Sets

Particularly, a simplicial object in **Sets**, the category of sets, will be called a simplicial set.

Definition 2.2.4 A *simplicial set* is a functor $K : \Delta^{op} \longrightarrow \mathbf{Sets}$ with the usual simplicial properties. A *simplicial map* is a natural transformation $f : K \longrightarrow L$ satisfying all the necessary simplicial properties.

The resulted category will be denoted by **SimpSets** or simply $\mathcal{S}$.

For a given two simplicial maps of simplicial sets, a homotopy between them can also be presented in the following definition.

Definition 2.2.5 Suppose $f, g : K \longrightarrow L$ are simplicial maps of simplicial sets. Then $f \simeq g$ if and only if there is a simplicial map

$$F : K \times \Delta[1] \longrightarrow L$$

with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in K$.

For every $n \geq 0$, define the *standard n-simplex* $\Delta[n] = hom(-, [n])$. Then, for each i, $0 \leq i \leq n$, define a subobject $\Lambda^i[n]$ of $\Delta[n]$ by

$$\Lambda^i[n]_m = \begin{cases} \Delta[n]_m & 0 \leq m \leq n-1 \\ \Delta[n]_{n-1} - \{\delta_n^i\} & m = n-1 \\ \text{all } m\text{-simplices degenerate} & m \geq n. \end{cases}$$

Intuitively, $\Lambda^i[n]$ is a box formed from an n-simplex by throwing away its i^{th} face and also its interior, i.e. the unique non-degenerate n-simplex $id_{[n]}$ in $\Delta[n]$.

Definition 2.2.6 An *(n, i)-box* in a simplicial set K is a simplicial map

$$\alpha : \Lambda^i[n] \longrightarrow K$$

and a *filler* for α is a simplicial map

$$\beta : \Delta[n] \longrightarrow K$$

which restricts to α on $\Lambda^i[n]$.

This leads to the definition of Kan complexes.

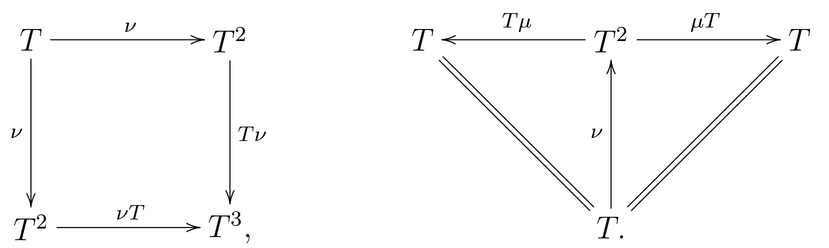
Definition 2.2.7 A simplicial set K satisfies the *(n,i)-extension condition (of Kan)* if any (n,i)-box in K has a filler. K is a *weak Kan complex* if it satisfies the (n,i)-extension condition for all pair (n,i) with $0 < i < n$, and it is a *Kan complex* if it satisfies the (n,i)-extension condition for all pair (n,i) with $0 \leq i \leq n$.

Thus a simplicial set K is *Kan* if all boxes have fillers.

2.3 Comonad Resolutions

We introduce the theory of comonads on a small category and the related simplicial objects, cf. Mac Lane [38].

Definition 2.3.1 A *comonad* (T, μ, ν) in a category $\mathbf{A}$ consists of a functor $T : \mathbf{A} \longrightarrow \mathbf{A}$ and two natural transformations, the *counit* $\mu : T \longrightarrow Id_{\boldsymbol{A}}$ and the *comultiplication* $\nu : T \longrightarrow T^2$ which make the following diagrams commute



Then for any object A of $\mathbf{A}$, we form a *simplicial object* (TA) "resolving" A as $(TA)_n = T^{n+1}A$ with the face and degeneracy operators

$$d_i = T^i \mu T^{n-i} : (TA)_n \longrightarrow (TA)_{n-1}$$
$$s_i = T^i \nu T^{n-i} : (TA)_n \longrightarrow (TA)_{n+1}.$$

2.3.1 Example in Cat

One application of comonad resolution theory comes from a system of the forgetful functor $U : \mathsf{Cat} \longrightarrow \mathsf{Grph}$ and the free functor $F : \mathsf{Grph} \longrightarrow \mathbf{Cat}$, where Cat is the category of small categories and Grph is the category of (unreflexive) directed graphs, cf. Dwyer and Kan [25] and Cordier [12].

Definition 2.3.2 The comonad (F, ϕ, ψ) defined on Cat consist of: for each small category $\mathbf{A}$, the objects of $F(\mathbf{A})$ the same objects as in $\mathbf{A}$, and a morphism from A to B in $F(\mathbf{A})$ being a string $(f_0, ..., f_n)$ of composable non-identity maps in $\mathbf{A}$ with domain $f_0 = A$ and codomain $f_n = B$. The *counit*

$$\phi : F(\boldsymbol{A}) \longrightarrow \boldsymbol{A},$$
$$(f_0, ..., f_n) \mapsto f_n f_{n-1} ... f_0$$

composes the maps in the string, and the *comultiplication*

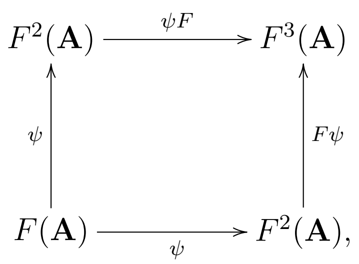
$$\psi : F(\boldsymbol{A}) \longrightarrow F^2(\boldsymbol{A}),$$
$$(f_0, ..., f_n) \mapsto ((f_0), (f_1), ..., (f_n)).$$

considers each map in a string as a string of length 1.

These satisfy co-associativity rules

$$\psi F.\psi = F\psi.\psi : F(\mathbf{A}) \longrightarrow F^3(\mathbf{A}),$$

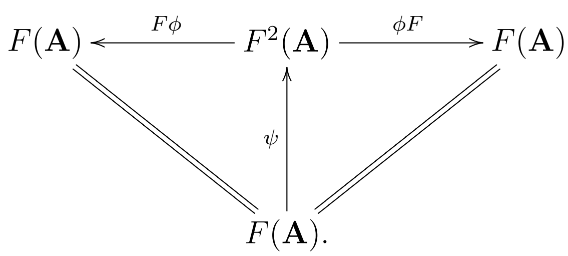
diagrammatically described by



and the triangle rules

$$\phi F.\psi = 1_F = F\phi.\psi : F(\mathbf{A}) \longrightarrow F(\mathbf{A}),$$

described by



Looking to the above definition, observe that in the lowest level we got two possible ways of forming maps from $F^2(\mathbf{A})$ to $F(\mathbf{A})$, and only one possible way of doing it inversely. For the next level, we got three possible maps from $F^3(\mathbf{A})$ to $F^2(\mathbf{A})$, and two possible maps inversely. These fit the definition of a simplicial object $\widetilde{F}(\mathbf{A})$ in Cat by defining $\widetilde{F}(\mathbf{A})_n = F^{n+1}(\mathbf{A})$, for $n = 0, 1, 2, ...$, to form

$$\cdots \begin{array}{c}\longrightarrow\\[-8pt]\longrightarrow\\[-8pt]\longrightarrow\\[-8pt]\longrightarrow\end{array} F(\mathbf{A})_2 \;\overset{\phi F^2, F\phi F, F^2\phi}{\begin{array}{c}\longrightarrow\\[-8pt]\longrightarrow\\[-8pt]\longrightarrow\end{array}}\; F(\mathbf{A})_1 \;\overset{\phi F, F\phi}{\begin{array}{c}\longrightarrow\\[-8pt]\longrightarrow\end{array}}\; F(\mathbf{A})_0.$$

Definition 2.3.3 $\widetilde{F}(\mathbf{A})$ is a simplicial object in Cat having

$$F(\mathbf{A})_n = F^{n+1}(\mathbf{A})$$

with face and degeneracy operators

$$\begin{aligned} d_i &= F^i\phi F^{n-i} : \widetilde{F}(\mathbf{A})_n \longrightarrow \widetilde{F}(\mathbf{A})_{n-1}, \ \text{ for } 0 \leq i \leq n, \\ s_i &= F^i\phi F^{n-i} : \widetilde{F}(\mathbf{A})_{n-1} \longrightarrow \widetilde{F}(\mathbf{A})_n, \ \text{ for } 0 \leq i \leq n. \end{aligned}$$

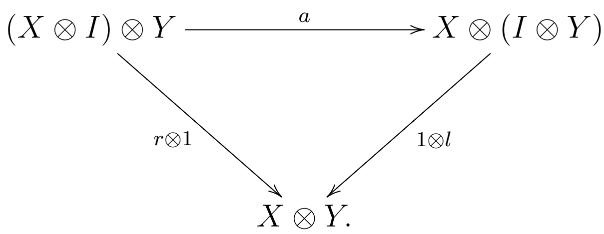
This material will be use later in Subsection 2.4.4 on the $\mathcal{S}$-category $\mathbb{S}(\mathbf{A})$.

2.4 V-Categories

For any monoidal category $\mathbf{V}$, there is an extensive theory of categories enriched over it, called $\mathbf{V}$-categories, for more details see Kelly [35].

Definition 2.4.1 A monoidal category $\mathbf{V}$=$(V_0, \otimes, I, a, l, r)$ consists of a category V_0, a functor $\otimes : V_0 \times V_0 \longrightarrow V_0$, an object I of V_0, and natural isomorphisms $a : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$, $l : I \otimes X \longrightarrow X$, and $r : X \otimes I \longrightarrow X$, subject to two coherence axioms expressing the commutativity of the following diagrams

$$\begin{array}{ccccc} ((W \otimes X) \otimes Y) \otimes Z & \xrightarrow{a} & (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{a} & W \otimes (X \otimes Y \otimes Z)) \\ {\scriptstyle a\otimes 1}\downarrow & & & & \uparrow{\scriptstyle 1\otimes a} \\ (W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow[a]{\hspace{6em}} & & W \otimes ((X \otimes Y) \otimes Z), \end{array}$$



The definition of **V**-category then is as follows.

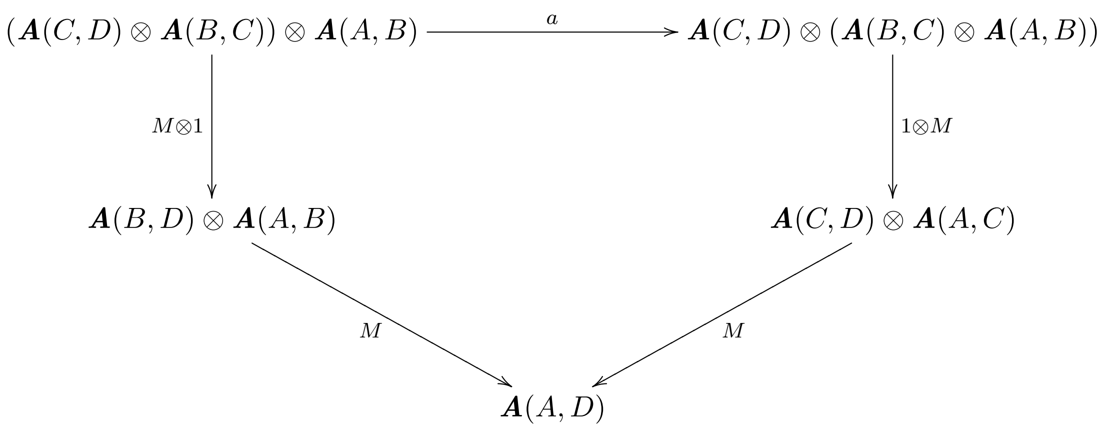
Definition 2.4.2 A **V**-category is a category **A** consists of a set $ob(\mathbf{A})$ of objects; a hom-object $\mathbf{A}(A, B) \in V_0$, for each pair of objects (A, B) of **A**; a composition law

$$M : \mathbf{A}(B, C) \otimes \mathbf{A}(A, B) \longrightarrow \mathbf{A}(A, C),$$

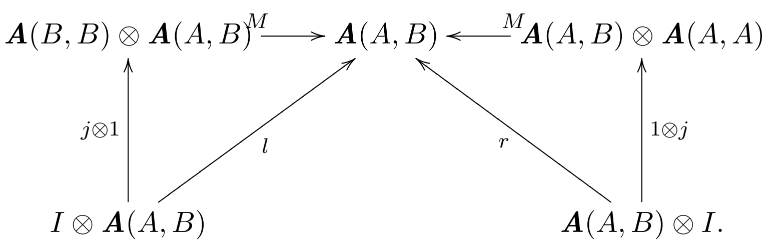
for each triple of objects (A, B, C); and an identity element

$$j : I \longrightarrow \boldsymbol{A}(A, A),$$

for each object A. These all subject to the associativity and unit axioms expressed by the commutativity of



and



For any two **V**-categories, the definition of a **V**-functor is as follows.

Definition 2.4.3 Suppose **A** and **B** are **V**-categories. A **V**-functor $T : \mathbf{A} \longrightarrow \mathbf{B}$ consists of a function $T : ob(\mathbf{A}) \longrightarrow ob(\mathbf{B})$ and, for each pair $A, B \in ob(\mathbf{A})$, a morphism $T_{AB} : \mathbf{A}(A,B) \longrightarrow \mathbf{B}(TA,TB)$, subject to compatibility with composition and with the identities expressed by the commutativity of

$$\begin{array}{ccc} \mathbf{A}(B,C)\otimes\mathbf{A}(A,B) & \xrightarrow{\ M\ } & \mathbf{A}(A,C) \\ {\scriptstyle T\otimes T}\downarrow & & \downarrow{\scriptstyle T} \\ \mathbf{B}(TB,TC)\otimes\mathbf{B}(TA,TB) & \xrightarrow[\ M\ ]{} & \mathbf{B}(TA,TC) \end{array}$$

and

$$\begin{array}{ccc} & & \mathbf{A}(A,A) \\ & \nearrow^{j} & \\ I & & \downarrow{\scriptstyle T} \\ & \searrow_{j} & \\ & & \mathbf{B}(TA,TA). \end{array}$$

For **V**=**Top**, we get **Top**-categories and **Top**-functors, cf. Boardman and Vogt [4]. We will need these **Top**-categories in the theory of **Top**-enriched homotopy coherent diagrams, discuss in Chapter 3. Additional to the above definition, we give the notion of **V**-natural transformation.

Definition 2.4.4 Suppose $T, S : \mathbf{A} \longrightarrow \mathbf{B}$ are **V**-functors. A **V**-natural transformation $\alpha : T \longrightarrow S$ is an $ob(\mathbf{A})$-indexed family of components $\alpha_A : I \longrightarrow \mathbf{B}(TA,SA)$ satisfying the **V**-naturality condition expressed by the commutativity of

$$\begin{array}{ccccc} & I\otimes\mathbf{A}(A,B) & \xrightarrow{\ \alpha_B\otimes T\ } & \mathbf{B}(TB,SB)\otimes\mathbf{B}(TA,TB) & \\ {\scriptstyle l^{-1}}\nearrow & & & & \searrow{\scriptstyle M} \\ \mathbf{A}(A,B) & & & & \mathbf{B}(TA,SB) \\ {\scriptstyle r^{-1}}\searrow & & & & \nearrow{\scriptstyle M} \\ & \mathbf{A}(A,B)\otimes I & \xrightarrow[\ S\otimes\alpha_A\ ]{} & \mathbf{B}(SA,SB)\otimes\mathbf{B}(TA,SA). & \end{array}$$

We explore further another important example of $\mathbf{V}$-categories, that is, $\mathcal{S}$-categories.

2.4.1 $\mathcal{S}$-Categories

For the case where $\mathbf{V} = \mathcal{S}$ and $\otimes$ is $\times$, we will have the following equivalent definition, cf. Quillen [48], Bousfield and Kan [7], and Kamps and Porter [34].

Definition 2.4.5 Let $\mathbf{A}$ be a category. We say that $\mathbf{A}_{\mathcal{S}}$ is an $\mathcal{S}$-category if:
(i) there exists a functor

$$\mathcal{A}(-,-) : \mathbf{A}^{op} \times \mathbf{A} \longrightarrow \mathcal{S}$$

together with a natural isomorphism

$$\begin{aligned} \mathbf{A}(A,B) &\longrightarrow \mathcal{A}(A,B)_0 \\ a &\mapsto \tilde{a}, \end{aligned}$$

the set of zero simplices of $\mathcal{A}(A,B)$; (this gives the identity morphisms of the category $\mathbf{A}$ as vertices of the various $\mathcal{A}(A,A)$ and hence a simplicial map, the identity label

$$Id_A : \Delta[0] \longrightarrow \mathcal{A}(A,A),$$

'naming' that vertex as being the identity on A; this level of formality will often be dropped but is useful to have behind the more lax presentation for when it is needed);
(ii) for each triple A, B, C of objects of $\mathbf{A}$, there is a associative composition morphism in $\mathcal{S}$

$$\begin{aligned} d^B_{A,C} : \mathcal{A}(A,B) \times \mathcal{A}(B,C) &\longrightarrow \mathcal{A}(A,C) \\ (f,g) &\mapsto gf; \end{aligned}$$

(iii) for $a \in \mathbf{A}(A,B)$, $f \in \mathcal{A}(B,C)_n$, $g \in \mathcal{A}(D,A)_n$ and on denoting by

$$\sigma^n : [n] \longrightarrow [0]$$

the unique such morphism in Δ, one has

$$\mathcal{A}(a, C)_n(f) = f\mathcal{A}(A, B)_{\sigma^n}(\tilde{a})$$

and

$$\mathcal{A}(D, a)_n(g) = \mathcal{A}(A, B)_{\sigma^n}(\tilde{a})g.$$

Cordier and Porter [14] then introduced the notion of locally Kan $\mathcal{S}$-category.

Definition 2.4.6 An $\mathcal{S}$-category $\mathbf{A}_{\mathcal{S}}$ is locally Kan if for each pair of objects A, B of $\mathbf{A}$, the simplicial set $\mathcal{A}(A, B)$ is a Kan complex.

Given two $\mathcal{S}$-categories, we can define an $\mathcal{S}$-functor between them as follows. This is equivalent to that in Definition 2.4.3 for the case $(\mathbf{V}, \otimes) = (\mathcal{S}, \times)$.

Definition 2.4.7 An $\mathcal{S}$-functor $T : \mathbf{A}_{\mathcal{S}} \longrightarrow \mathbf{B}_{\mathcal{S}}$ consists of a function $T : ob(\mathbf{A}) \longrightarrow ob(\mathbf{B})$ and for each pair $A, B \in ob(\mathbf{A})$, a simplicial map

$$T_{AB} : \mathcal{A}(A, B) \longrightarrow \mathcal{B}(TA, TB)$$

such that if A, B, C are objects in $\mathbf{A}$, the diagrams

$$\begin{array}{ccc} \mathcal{A}(A, B) \times \mathcal{A}(B, C) & \xrightarrow{comp} & \mathcal{A}(A, C) \\ \Big\downarrow{\scriptstyle T_{A,B} \times T_{B,C}} & & \Big\downarrow{\scriptstyle T_{A,C}} \\ \mathcal{B}(TA, TB) \times \mathcal{B}(TB, TC) & \xrightarrow[comp]{} & \mathcal{B}(TA, TC) \end{array}$$

and

$$\begin{array}{ccc} & & \mathcal{A}(A, A) \\ & \nearrow^{Id_A} & \\ \Delta[0] & & \Big\downarrow{\scriptstyle T_{A,A}} \\ & \searrow_{Id_{TA}} & \\ & & \mathcal{B}(TA, TA). \end{array}$$

It is obvious then to have the notion of $\mathcal{S}$-natural transformations (cf. Definiton 2.4.4 for an equivalent form).

Definition 2.4.8 Suppose $T, S : \mathbf{A}_{\mathcal{S}} \longrightarrow \mathbf{B}_{\mathcal{S}}$ are $\mathcal{S}$-functors. An $\mathcal{S}$-natural transformation $\alpha : T \longrightarrow S$ is an $ob(\mathbf{A})$-indexed family of components $\alpha_A : \Delta[0] \longrightarrow \mathcal{B}(TA, SA)$ satisfying the $\mathcal{S}$-naturality condition expressed by the commutativity of

$$\begin{array}{ccccc} & \mathcal{A}(A,B)\times\Delta[0] & \xrightarrow{T_{A,B}\times\alpha_B} & \mathcal{B}(TA,TB)\times\mathcal{B}(TB,SB) & \\ {}^{l^{-1}}\nearrow & & & & \searrow^{comp} \\ \mathcal{A}(A,B) & & & & \mathbf{B}(TA,SB) \\ {}_{r^{-1}}\searrow & & & & \nearrow_{comp} \\ & \Delta[0]\times\mathcal{A}(A,B) & \xrightarrow[\alpha_A\times S_{A,B}]{} & \mathcal{B}(TA,SA)\times\mathcal{B}(SA,SB). & \end{array}$$

Examples of $\mathcal{S}$-categories are numerous, we will look at $\mathbf{Top}_{\mathcal{S}}$, $\mathcal{S}_{\mathcal{S}}$ and $\mathbb{S}(\mathbf{A})$.

2.4.2 The $\mathcal{S}$-Category $\mathbf{Top}_{\mathcal{S}}$

We gather information on $\mathbf{Top}_{\mathcal{S}}$ from Quillen [48].

Definition 2.4.9 Suppose X and Y are spaces. Define $\mathbf{Top}_{\mathcal{S}}(X, Y)$ by

$$\mathbf{Top}_{\mathcal{S}}(X, Y)_n = \mathbf{Top}(X \times \Delta[n], Y)$$

where $\Delta[n]$ is a standard n-simplex. For $f \in \mathbf{Top}(X \times \Delta[n], Y)$ and $g \in \mathbf{Top}(Y \times \Delta[n], Z)$, the composite map gf is given by

$$X \times \Delta[n] \xrightarrow{id\times diag} X \times \Delta[n] \times \Delta[n] \xrightarrow{f\times id} Y \times \Delta[n] \xrightarrow{g} Z.$$

The resulting $\mathcal{S}$-category is denoted $\mathbf{Top}_{\mathcal{S}}$.

Theorem 2.4.10 $\mathbf{Top}_{\mathcal{S}}$ is a locally Kan $\mathcal{S}$-category.

2.4.3 The $\mathcal{S}$-category $\mathcal{S}_{\mathcal{S}}$

The category $\mathcal{S}$ is an $\mathcal{S}$-category enriched over itself.

Definition 2.4.11 Suppose K and L are simplicial sets. Define $\mathcal{S}_{\mathcal{S}}(K, L)$ by

$$\mathcal{S}_{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L).$$

If $f \in \mathcal{S}(K \times \Delta[n], L)$ and $g \in \mathcal{S}(L \times \Delta[n], M)$, the composite map gf is given by

$$K \times \Delta[n] \xrightarrow{id \times diag} K \times \Delta[n] \times \Delta[n] \xrightarrow{f \times id} L \times \Delta[n] \xrightarrow{g} M.$$

The resulting $\mathcal{S}$-category is denoted by $\mathcal{S}_{\mathcal{S}}$.

Theorem 2.4.12 $\mathcal{S}_{\mathcal{S}}$ is an $\mathcal{S}$-category.

It is well known that $\mathcal{S}_{\mathcal{S}}$ is not locally Kan, cf. Kamps and Porter [34].

For the $\mathcal{S}$-category $\mathcal{S}_{\mathcal{S}}$, there is an $\mathcal{S}$-functor

$$\begin{aligned} \pi_0 : \mathcal{S}_{\mathcal{S}} &\longrightarrow \mathbf{Ho}(\mathcal{S}) \\ (K, L) &\mapsto [K, L], \end{aligned}$$

where $[K, L]$ denote the homotopy class of simplicial maps, and $\mathbf{Ho}(\mathcal{S})$ is the homotopy category of simplicial sets.

2.4.4 The $\mathcal{S}$-category $\mathbb{S}(\mathbf{A})$

It is from the earlier ideas of Dwyer and Kan [25] that Cordier [12] and, Porter and Cordier [47] introduced the notion of $\mathcal{S}$-category $\mathbb{S}(\mathbf{A})$, which comes from the comonad resolution on a small category $\mathbf{A}$.

Referring to the previous Subsection 2.3.1, we already know that there is a comonad (F, ϕ, ψ) defined on Cat. This then will give us, for each small category $\mathbf{A}$, a simplicial object $\widetilde{F}(\mathbf{A})$. Generally, it is not true that a simplicial object in Cat gives an $\mathcal{S}$-category, however in this case, the corresponding simplicial object $\widetilde{F}(\mathbf{A})$ does gives one an $\mathcal{S}$-category $\mathbb{S}(\mathbf{A})$. The corresponding object set is defined by $ob(\mathbb{S}(\mathbf{A})) = ob(\mathbf{A})$, and the morphisms by

$$\mathbb{S}(\mathbf{A})(A, B) = \widetilde{F}(\mathbf{A})(A, B),$$

for each A, B objects of $\mathbf{A}$. An element of $\mathbb{S}(\mathbf{A})(A, B)_n$ is a string of strings of strings ... of strings of non-identity maps of $\mathbf{A}$, together with information on how the substrings are bracketted together. The composition is just juxtaposition of these bracketted strings

$$(f_0, ..., f_m)(f_{m+1}, ..., f_n) = (f_0, ..., f_n).$$

We will call $\mathbb{S}(\mathbf{A})$ the *Dwyer-Kan-Cordier (DKC) $\mathcal{S}$-category on $\mathbf{A}$*. Furthermore, there is an *augmentation functor*

$$aug : \mathcal{S}(\mathbf{A}) \longrightarrow \mathbf{A}$$

which for each pair (A, B) of objects of $\mathbf{A}$ gives a homotopy equivalence of simplicial set $\mathcal{S}(\mathbf{A})(A, B) \longrightarrow aug(\mathbf{A}(A, B), 0)$, where $aug(\mathbf{A}(A, B), 0)$ is the simplicial set with $aug(\mathbf{A}(A, B), 0)_0 = \mathbf{A}(A, B)$ and all n-simplexes with $n > 0$ degenerate.

The illustration of the above construction, extends those in Cordier [12] and Porter and Cordier [47], will be demonstrated using the following three consecutive examples. All of them will be applied later for the construction of the DKC $\mathcal{S}$-categories in Chapters 4, 5 and 6.

Let [2] be the category whose objects are numerals $\{0, 1, 2\}$ and whose morphisms are $01 : 0 \longrightarrow 1$, $02 : 0 \longrightarrow 2$ and $12 : 1 \longrightarrow 2$. Then we will have $\mathbb{S}[2]$ and specify all non-degenerate simplices. We use "," to separate the basic morphisms and ";" to separate composites of morphisms.

Dimension 0

$$\begin{aligned}
\mathbb{S}[2](0,1)_0 &= \{(01)\} \\
\mathbb{S}[2](0,2)_0 &= \{(02); (01, 12)\} \\
\mathbb{S}[2](1,2)_0 &= \{(12)\}
\end{aligned}$$

Dimension 1

$$\begin{aligned}\mathbb{S}[2](0,1)_1 &= \{((01))\}\\ \mathbb{S}[2](0,2)_1 &= \{((02));((01,12));((01),(12))\}\\ \mathbb{S}[2](1,2)_1 &= \{((12))\}\end{aligned}$$

It is worthwhile noting that $d_0 = \phi F$ and $d_1 = F\phi$, so that for instance

$$\begin{aligned}d_0((01,12)) &= (01,12),\\ d_1((01,12)) &= (02).\end{aligned}$$

Thus d_0 composes within outer brackets and then removes them, and d_1 composes within inner brackets and then removes them. These can be presented by a diagram

$$(02) \xrightarrow{((01),(12))} (01,12).$$

Note that $\mathbb{S}[2](0,2) \cong \Delta[1]$.

Next, the category [3] whose objects are numerals $\{0,1,2,3\}$ and whose morphisms are $01 : 0 \longrightarrow 1$, $02 : 0 \longrightarrow 2$ and so on, the properties of $\mathbb{S}[3]$ can be demonstrate as in the following.

Dimension 0

$$\begin{aligned}\mathbb{S}[3](0,1)_0 &= \{(01)\}\\ \mathbb{S}[3](0,2)_0 &= \{(02);(01,12)\}\\ \mathbb{S}[3](0,3)_0 &= \{(03);(01,13);(02,23);(01,12,23)\}\\ \mathbb{S}[3](1,2)_0 &= \{(12)\}\\ \mathbb{S}[3](1,3)_0 &= \{(13);(12,23)\}\\ \mathbb{S}[3](2,3)_0 &= \{(23)\}\end{aligned}$$

Dimension 1

$$\mathbb{S}[3](0,1)_1 = \{((01))\}$$

$$\begin{aligned}
\mathbb{S}[3](0,2)_1 &= \{((02));((01,12));((01),(12))\} \\
\mathbb{S}[3](0,3)_1 &= \{((03));((01,13));((01),(13));((02,23));((02),(23)); \\
&\quad ((01,12,23));((01),(12,23));((01,12),(23));((01),(12),(23))\} \\
\mathbb{S}[3](1,2)_1 &= \{((12))\} \\
\mathbb{S}[3](1,3)_1 &= \{((13));((12,23));((12),(23))\} \\
\mathbb{S}[3](2,3)_1 &= \{((23))\}
\end{aligned}$$

Dimension 2

$$\begin{aligned}
\mathbb{S}[3](0,1)_2 &= \{(((01)))\} \\
\mathbb{S}[3](0,2)_2 &= \{(((02)));(((01,12)));(((01),(12)));(((01)),((12)))\} \\
\mathbb{S}[3](0,3)_2 &= \{(((03)));(((01,13)));(((01),(13)));(((01)),((13))); \\
&\quad (((02,23)));(((02),(23)));(((02)),((23)));(((01,12,23))); \\
&\quad (((01),(12,23)));(((01)),((12,23)));(((01,12),(23))); \\
&\quad (((01),(12),(23)));(((01,12)),((23)));(((01),(12),(23))); \\
&\quad (((01)),((12),(23)));(((01),(12)),((23)));(((01)),((12)),((23)))\} \\
\mathbb{S}[3](1,2)_2 &= \{(((12)))\} \\
\mathbb{S}[3](1,3)_2 &= \{(((13)));(((12,23)));(((12),(23)));(((12)),((23)))\} \\
\mathbb{S}[3](2,3)_2 &= \{(((23)))\}
\end{aligned}$$

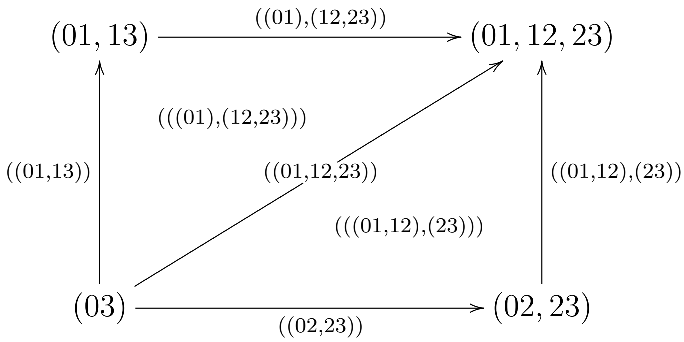
Analogously, $d_0 = \phi F^2$, $d_1 = F\phi F$ and $d_2 = F^2\phi$, so that for instance

$$\begin{aligned}
d_0(((01),(12,23))) &= ((01),(12,23)), \\
d_1(((01),(12,23))) &= ((01,12,23)), \\
d_2(((01),(12,23))) &= ((01),(13)),
\end{aligned}$$

where d_0 composes within outer brackets and then removes them, d_1 composes within 1^{st}-inner brackets and then removes them, and d_2 composes within inner brackets and then removes them. For example:

$$d_0(((01,12),(23))) = ((01,12),(23)),$$
$$d_1(((01,12),(23))) = ((01,12,23)),$$
$$d_2(((01,12),(23))) = ((02),(23)).$$

The non-degenerate 2-simplexes in $\mathbb{S}[3](0,3)$ can both be presented by a square-face diagram



Note that $\mathbb{S}[3](0,3) \cong \Delta[1]^2$. Some of the compositions are:

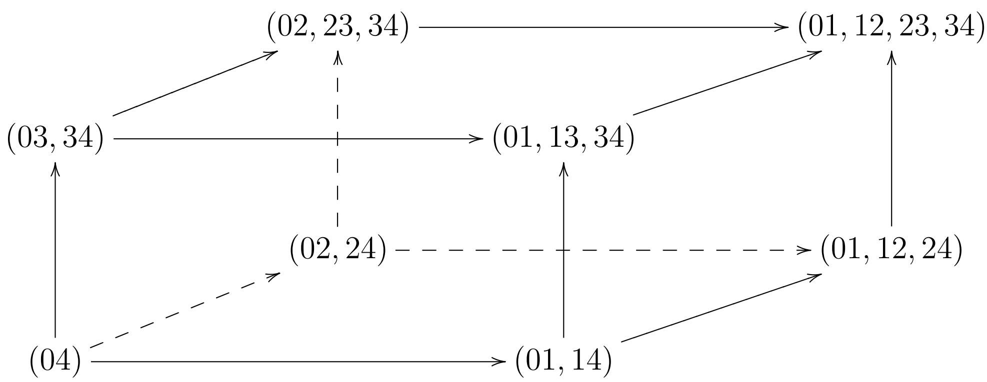
$$\mathcal{S}[3](0,1)_1 \times \mathbb{S}[3](1,3)_1 \longrightarrow \mathbb{S}[3](0,3)_1$$

such that $\Big(((01)); ((12,23))\Big) \mapsto ((01),(12,23))$,

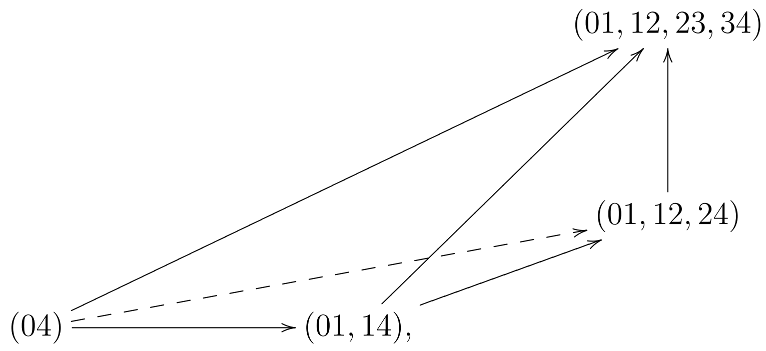
$$\mathcal{S}[3](0,2)_1 \times \mathbb{S}[3](2,3)_1 \longrightarrow \mathbb{S}[3](0,3)_1$$

such that $\Big(((01,12)); ((23))\Big) \mapsto ((01,12),(23))$, etc, where ";" now uses to seperates between different elements.

For $\mathbb{S}[4]$, we will have up to dimension 3, that the properties of dimensions 0, 1 and 2 are similar to those in the above examples. This particular case however will produce a cube face-diagram



containing six 3-simplexes. They are:
(1) The 3-simplex $((((01)),((12),(23,34))))$ presented by

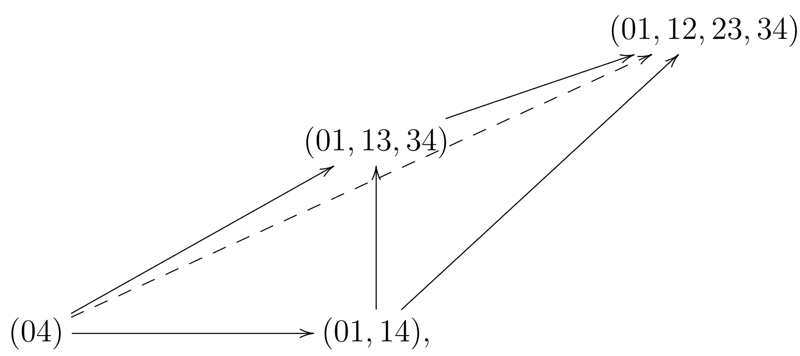


with

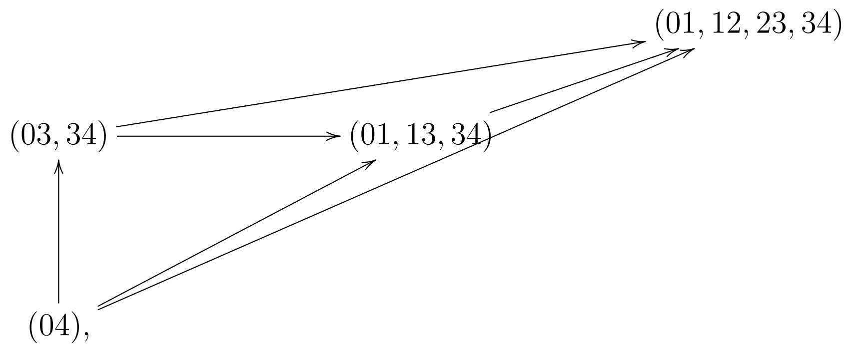
$$\begin{aligned} d_0((((01)),((12),(23,34)))) &= (((01)),((12),(23,34))), \\ d_1((((01)),((12),(23,34)))) &= (((01),(12),(23,34))), \\ d_2((((01)),((12),(23,34)))) &= (((01),(12,23,34))), \\ d_3((((01)),((12),(23,34)))) &= (((01),(12,24))). \end{aligned}$$

Thus, d_0 composes within outer brackets and then removes them, d_1 composes within 1^{st}-inner brackets and then removes them, d_2 composes within 2^{nd}-inner brackets and then removes them, and d_3 composes within inner brackets and the removes them. Similar processes also apply to the other five cases below, namely:

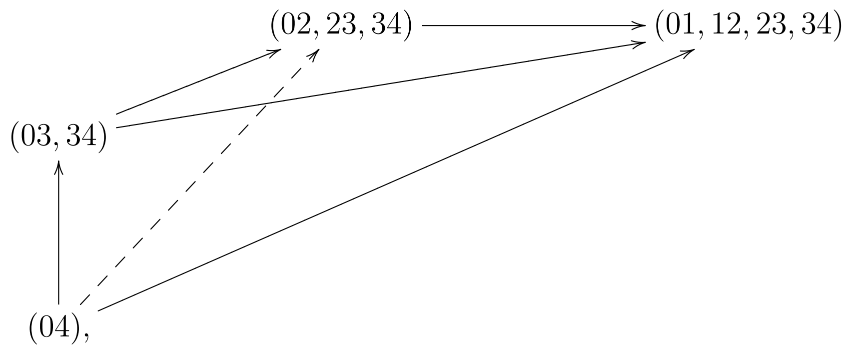
(2) The 3-simplex $((((01)),((12,23),(34))))$ presented by



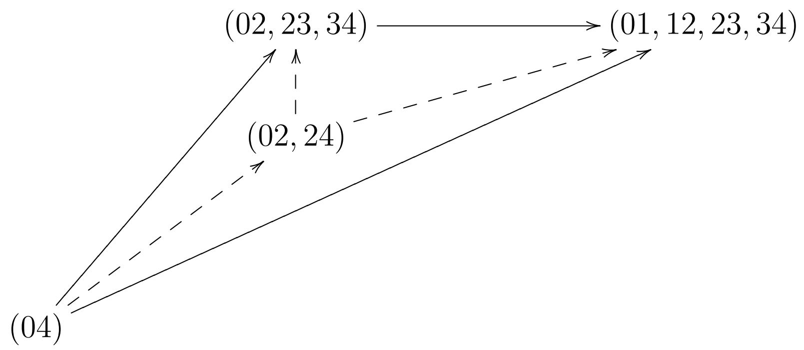
(3) The 3-simplex $((((01),(12,23)),((34))))$ presented by



(4) The 3-simplex ((((01, 12), (23)), ((34)))) presented by

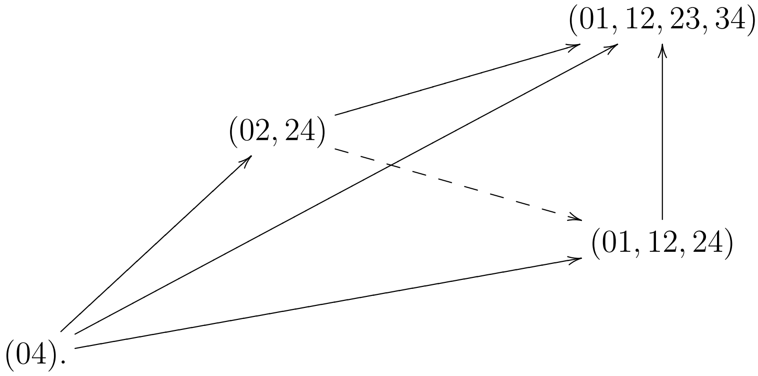


(5) The 3-simplex ((((01, 12)), ((23), (34)))) presented by



and

(6) The 3-simplex ((((01), (12)), ((23, 34)))) presented by



Note also that $\mathbb{S}[4](0,4) \cong \Delta[1]^3$.

Generally, we will have $\mathbb{S}[n](0,n) \cong \Delta[1]^{n-1}$, cf. Cordier and Porter [13].

2.5 Coskeleta Theory

For benefits of the later works in Chapter 6, we give some backgrounds on coskeleta theory, cf. Quillen [48], Artin and Mazur [3] and Duskin [23], [24].

2.5.1 Truncated Simplicial Objects

Suppose $\Delta_{n]}$ designates the full subcategory of Δ determined by those sets whose cardinality is at most $n+1$.

Definition 2.5.1 A *n-truncated simplicial object* is a contravariant functor $X_{tr} : \Delta_{n]}^{op} \longrightarrow \mathbf{C}$, represented by

$$X_n \underset{d_0}{\overset{d_n}{\underset{\vdots}{\rightrightarrows}}} X_{n-1} \underset{d_0}{\overset{d_{n-1}}{\underset{\vdots}{\rightrightarrows}}} X_{n-2} \underset{d_0}{\overset{d_{n-2}}{\underset{\vdots}{\rightrightarrows}}} \cdots \rightrightarrows X_2 \rightrightarrows X_1 \rightrightarrows X_0,$$

where d_i and s_i verify the simplicial identities whenever they are defined.

If we denote by $\mathbf{Tr}_{n]}\mathbf{Simp(C)}$ the category of n-truncated simplicial objects of $\mathbf{C}$, then the restriction functor induces a functor (truncation at level n)

$$tr_{n]} : \mathbf{Simp(C)} \longrightarrow \mathbf{Tr_{n]}Simp(C)}$$

which simply forgets that portion of a simplicial object which appears in dimensions higher than n.

2.5.2 Simplicial Kernels and n-Coskeletons

For any n-truncated simplicial object X_{tr}, the "possible boundaries of $(n+1)$-simplices" developed over it is called a simplicial kernel.

Definition 2.5.2 Suppose $\mathbf{C}$ has finite inverse limits and X_{tr} is a n-truncated simplicial object in $\mathbf{C}$. A *simplicial kernel* of the family of face operators $\{d_0, ..., d_n\}$ consists of an object K_{n+1} with $n+2$ face operators $\{p_0, ..., p_{n+1}\}$,

$$K_{n+1} \underset{p_0}{\overset{p_{n+1}}{\underset{\vdots}{\rightrightarrows}}} X_n \underset{d_0}{\overset{d_n}{\underset{\vdots}{\rightrightarrows}}} X_{n-1} \underset{d_0}{\overset{d_{n-1}}{\underset{\vdots}{\rightrightarrows}}} \cdots$$

with the corresponding simplicial properties. Such family $\{p_0, ..., p_{n+1}\}$ has the following universal property: given any family $\{x_0, ..., x_{n+1}\}$ of $n+2$ arrows

$$T \underset{x_0}{\overset{x_{n+1}}{\underset{\vdots}{\rightrightarrows}}} X_n \underset{d_0}{\overset{d_n}{\underset{\vdots}{\rightrightarrows}}} X_{n-1} \underset{d_0}{\overset{d_{n-1}}{\underset{\vdots}{\rightrightarrows}}} \cdots$$

which satisfy the simplicial equalities with the face operators $\{d_0, ..., d_n\}$ of the n-simplex of X_{tr}, there exists a unique arrow $x : T \longrightarrow K_{n+1}$ such that $p_i x = x_i$.

With the simplicial kernel, define then the *degeneracy operators*

$$K_{n+1} \underset{q_n}{\overset{q_0}{\underset{\vdots}{\leftleftarrows}}} X_n \cdots$$

as follows: for each j, $0 \leq j \leq n$, the family $\{\alpha_{n+1,j}, ..., \alpha_{1,j}, \alpha_{0,j}\}$ given by

$$\alpha_{ij} = \begin{cases} q_{j-1}d_i & i < j \\ Id & i = j, j+1 \\ q_j d_{i-1} & i > j+1 \end{cases}$$

is easily seen to satisfy the simplicial identities with the face operators $\{d_i\}$; hence there exists a unique $q_j : X_n \longrightarrow K_{n+1}$ such that $p_i q_j = \alpha_{ij}$.

The $(n+1)$-truncated simplicial object X^* so-defined has the following universal property: If T is any simplicial object and f is an n-truncated simplicial map from the n-truncation of T into the n-truncated simplicial object X_{tr}, then the arrow $f_{n+1} : T_{n+1} \longrightarrow K_{n+1}$ defined as a unique lifting of the family $\{f_n d_i\}_{0 \leq i \leq n+1}$, defines an $(n+1)$-truncated simplicial map from the $(n+1)$-truncation of T into the $(n+1)$-truncated simplicial object X^*, that is, it extends the n-truncated map by one level.

Definition 2.5.3 The *n-coskeleton* of n-truncated simplicial object X_{tr} is the iteration of successive simplicial kernels, that is, define $(Cosk_n X_{tr})_{n+1}$ as K_{n+1} together with its canonical face and degeneracy operators.

Thus for example, the (-1)-*coskeleton* will be just the constant simplex having the terminal object 1 in every dimension, while the 0-*coskeleton* will be the product simplex

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} X_0 \times X_0 \times X_0 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} X_0 \times X_0 \substack{\longrightarrow\\\longrightarrow} X_0.$$

Chapter 3

Homotopy Coherent Diagrams

3.1 Introduction

It is well known that the canonical projection functor from **Top** to **Ho(Top)**, the category of spaces and homotopy classes of maps, does not preserve limits and colimits. The same problem holds for the categories **Top***, of based spaces and based maps, and **Ho(Top*****)**. Therefore, when dealing with constructions involving homotopies, one often has to substitute limits and colimits by something else, and the homotopy limits and colimits are in many cases the spaces having the universal properties one wants. The definitions of homotopy limits and colimits rely heavily on the concept of **A**-diagram, where **A** is a **Top**- or $\mathcal{S}$-category, cf. Boardman and Vogt [4], and Vogt [55]. We discuss **A**-diagrams in Section 3.2.

One approach to studying homotopy coherent diagrams, which concerns us here in this thesis, considers the generalisation of homotopy limits and colimits. Vogt [55] used a functorial construction T on $\mathbb{T}$, the category of **Top**-categories and **Top**-functors, to itself. He defined, for each **Top**-category **A**, a homotopy **A**-diagram as a T**A**-diagram. This particular **Top**-enriched homotopy coherent diagram will be discussed briefly in Section 3.3.

Vogt's notion of homotopy **A**-diagram was simplified by Cordier [12] to become a homotopy coherent diagram of type **A** in **Top**. Cordier then realized that **Top** is in particular a (locally Kan) $\mathcal{S}$-category, denoted by $\mathbf{Top}_{\mathcal{S}}$, and showed that the above homotopy coherent diagram of type **A** in **Top** is equivalent to an $\mathcal{S}$-functor

$$\mathbb{S}(\mathbf{A}) \longrightarrow \mathbf{Top}_{\mathcal{S}},$$

where $\mathbb{S}(\mathbf{A})$ is the DKC $\mathcal{S}$-category on **A**. We discuss this idea of $\mathcal{S}$-enriched homotopy coherent diagram in Section 3.4.

Later, Cordier and Porter [13], [14] gave the definition of a generalized $\mathcal{S}$-enriched homotopy coherent diagram, i.e. of type **A** in $\mathbf{B}_{\mathcal{S}}$, an arbitrary (locally Kan) $\mathcal{S}$-category, as being an $\mathcal{S}$-functor

$$\mathbb{S}(\mathbf{A}) \longrightarrow \mathbf{B}_{\mathcal{S}}.$$

This generalisation will be explained in Section 3.5.

3.2 A-diagrams

We use Boardman and Vogt [4] as a reference for the results on **Top**-categories and **Top**-functors. For the general **V**-enriched categories, see Kelly [35]. The following statements are due to Vogt [55].

Definition 3.2.1 Let **A** be a **Top**-category. An **A**-diagram D consists of a function

$$D_0 : ob(\mathbf{A}) \longrightarrow ob(\mathbf{Top})$$

and a collection of continuous maps

$$D_{A,B} : \mathbf{A}(A, B) \times D_0A \longrightarrow D_0B,$$

for each pair of objects (A, B) of **A** such that:
(i) $D_{A,A}(id_A; x) = x$, for all $x \in D_0A$,
(ii) $D_{A,C}(gf; x) = D_{B,C}(g; D_{A,B}(f; x))$, for all $f : A \longrightarrow B$ and $g : B \longrightarrow C$.

The use of “;” in $D_{A,B}(f;x)$ here and in its later use enhances the point that x is of a different sort from f, the latter being, in general, a string of symbols separated by commas.

For a given two **A**-diagrams, a homomorphism between them is defined as follows.

Definition 3.2.2 Let $\mathbf{A}$ be a **Top**-category and D and E two **A**-diagrams. A homomorphism $f : D \longrightarrow E$ is a $(\mathbf{A} \times [1])$-diagram whose restriction to $\mathbf{A} \times \{(0)\}$ is D and to $\mathbf{A} \times \{(1)\}$ is E.

Equivalently, the above consists of a collection of based maps $f_A : D_0A \longrightarrow E_0A$, one for each $A \in ob(\mathbf{A})$, such that the following diagram

$$\begin{array}{ccc}
\mathbf{A}(A,B) \times D_0A & \xrightarrow{D_{A,B}} & D_0B \\
\downarrow{\scriptstyle id\times f_A} & & \downarrow{\scriptstyle f_B} \\
\mathbf{A}(A,B) \times E_0A & \xrightarrow[E_{A,B}]{} & E_0B
\end{array}$$

commutes.

For a given **Top**-functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ and a **B**-diagram D, there exists an induced **A**-diagram DF as follows.

Definition 3.2.3 Suppose $F : \mathbf{A} \longrightarrow \mathbf{B}$ is a **Top**-functor and let D be a **B**-diagram. An induced **A**-diagram DF by F is defined by

$$(DF)_0 = D_0F : ob(\mathbf{A}) \longrightarrow ob(\mathbf{Top}),$$
$$(DF)_{A,A'} = D_{FA,FA'}F : \mathbf{A}(A,A') \times D_0FA \longrightarrow D_0FA'.$$

3.3 Top-Enriched Homotopy Coherent Diagrams

Vogt's construction of what he called homotopy **A**-diagrams goes as follows.

Let $\mathbb{T}$ be the category of **Top**-categories and **Top**-functors. Define a functor

$$T : \mathbb{T} \longrightarrow \mathbb{T}$$

such that for each **Top**-category $\mathbf{A}$, we set

$$\mathbf{A}_n(A, B) = \{(f_n, f_{n-1}, ..., f_1) \in mor(\mathbf{A})^n : f_n...f_1 : A \longrightarrow B\} \text{ for } n > 0,$$

$$\mathbf{A}_0(A, B) = \begin{cases} id_A & \text{if } A = B \\ \emptyset & \text{if } A \neq B, \end{cases}$$

$ob(T\mathbf{A}) = ob(\mathbf{A})$, and

$$T\mathbf{A}(A, B) = \textstyle\coprod_{n \geq 0} \mathbf{A}_{n+1}(A, B) \times I^n,$$

where composition is given by

$$\begin{array}{l}(g_0, u_1, g_1, ..., g_m, u_m)(f_0, t_1, f_1, ..., f_n, t_n) = \\ (f_0, t_1, f_1, ..., f_n, t_n, 0, g_0, u_1, g_1, ..., g_m, u_m),\end{array}$$

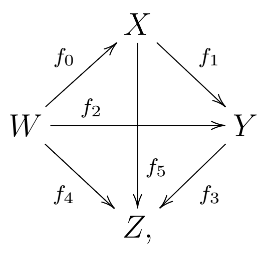
for each $((f_0, f_1, ..., f_n), (t_1, ..., t_n)) = (f_0, t_1, f_1, ..., f_n, t_n) \in \mathbf{A}_n(A, B)$ and $((g_0, g_1, ..., g_m), (u_1, ..., u_m)) = (g_0, u_1, g_1, ..., g_m, u_m) \in \mathbf{A}_m(B, C)$.

Definition 3.3.1 A homotopy $\mathbf{A}$-diagram is a $T\mathbf{A}$-diagram D such that $D_{A,B}(f_1, t_1, f_1, ..., f_n, t_n; x) =$

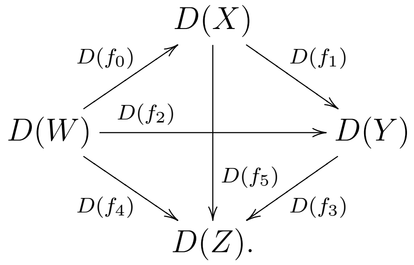
$$\begin{cases} D_{A,B}(f_1, t_1, f_2, ..., f_n, t_n; x) & \text{if } f_0 = id \\ D_{A,B}(f_0, t_1, f_1, ..., f_{i-1}, t_i t_{i+1}, f_{i+1}, ..., f_n, t_n; x) & \text{if } f_i = id, \ 0 < i < n \\ D_{A,B}(f_0, t_1, f_1, ..., f_{n-1}, t_{n-1}; x) & \text{if } f_n = id \\ D_{A,B}(f_0, t_1, f_1, ..., t_{i-1}, f_i f_{i-1}, t_{i+1}, ..., f_n, t_n; x) & \text{if } t_i = 1 \end{cases}$$

with $x \in D_0 A$ and $(f_0, t_1, f_1, ..., f_n, t_n) \in T\mathbf{A}(A, B)$.

We examine one of the simplest cases of the above definition. Let $\mathbf{A}$ be the category given by the commutative diagram



considered as a **Top**-category with discrete structure on the spaces of morphisms. Observe that, in order to satisfy the definition of a homotopy **A**-diagram, we have to construct the following possibly non-commutative diagram of spaces



Now, there are four 1-dimensional homotopy **A**-diagrams represented in the above diagram, namely:

$$D_{W,Y}(f_0, t_1, f_1; w) : T\mathbf{A}(W, Y) \times D_0W \longrightarrow D_0Y$$

which gives a homotopy $H(w, t_1) : D(W) \times I \longrightarrow D(Y)$;

$$D_{W,Z}(f_0, t_1, f_5; w) : T\mathbf{A}(W, Z) \times D_0W \longrightarrow D_0Z$$

which gives a homotopy $K(w, t_1) : D(W) \times I \longrightarrow D(Z)$;

$$D_{X,Z}(f_1, t_1, f_3; x) : T\mathbf{A}(X, Z) \times D_0X \longrightarrow D_0Z$$

which gives a homotopy $L(x, t_1) : D(X) \times I \longrightarrow D(Z)$; and

$$D_{W,Z}(f_2, t_1, f_3; w) : T\mathbf{A}(W, Z) \times D_0W \longrightarrow D_0Z$$

which gives a homotopy $M(w, t_1) : D(W) \times I \longrightarrow D(Z)$.

Further then, there are two 2-dimensional homotopy **A**-diagrams, namely:

$$D_{W,Z}(f_0, t_1, f_1, t_2, f_3; w) : T\mathbf{A}(W, Z) \times D_0W \longrightarrow D_0Z$$

which gives the first 2-dimensional homotopy

$$f_3HL(w, t_1, t_2) : D(W) \times I^2 \longrightarrow D(Z);$$

and

$$D_{W,Z}(f_0, t_1, f_1, t_2, f_3; w) : T\mathbf{A}(W, Z) \times D_0 W \longrightarrow D_0 Z$$

which gives the second 2-dimensional homotopy

$$M(f_0 \times id)K(w, t_1, t_2) : D(W) \times I^2 \longrightarrow D(Z).$$

All of the above informations can be presented in the following diagram

$$\begin{array}{ccc} f_3 f_2 & \xrightarrow{f_3 H} & f_3 f_2 f_1 \\ {\scriptstyle M}\uparrow & & \uparrow{\scriptstyle L(f_0 \times id)} \\ f_4 & \xrightarrow[K]{} & f_5 f_0. \end{array}$$

This example can easily be extended to the 3-dimensional homotopy **A**-diagrams.

If $f : \mathbf{A} \longrightarrow \mathbf{B}$ is a **Top**-functor, it is clear that a given homotopy **B**-diagram induces a homotopy **A**-diagram. The importance of this result will be seen later when we apply it to develop the notions of homotopy homomorphism of homotopy **A**-diagrams, simplicial homotopy of homotopy homomorphisms, and composite of homotopy homomorphisms.

Corollary 3.3.2 Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ be a **Top**-functor. If D is a homotopy **B**-diagram, i.e. a $T\mathbf{B}$-diagram, then DTF is a homotopy **A**-diagram, i.e. a $TF\mathbf{A}$-diagram, defined by

$$(DTF)_0 = D_0 TF : ob(\mathbf{A}) \longrightarrow ob(\mathbf{Top}),$$
$$(DTF)_{A,A'} = D_{TFA,TFA'} TF : T\mathbf{A}(A, A') \times D_0 TFA \longrightarrow D_0 TFA'.$$

3.3.1 Homotopy Homomorphisms

Recall that Δ is the category whose objects are the non-empty finite totally ordered sets $[n] = \{0, 1, ..., n\}$, where n are integers, and in which the morphisms are monotonic increasing maps $\mu : [m] \longrightarrow [n]$ such that $i \leq j$ implies $\mu(i) \leq \mu(j)$. Let $\mathbb{T}$ be a category of **Top**-categories and **Top**-functors, and

$T : \mathbb{T} \longrightarrow \mathbb{T}$ be a functor from $\mathbb{T}$ to itself.

For any **Top**-category $\mathbf{A}$ in $\mathbb{T}$, one then can generate a cosimplicial object

$$\cdots \; (\mathbf{A} \times [2]) \underset{id\times\delta_2^0}{\overset{id\times\delta_2^2}{\longleftarrow}} (\mathbf{A} \times [1]) \underset{id\times\delta_1^0}{\overset{id\times\delta_1^1}{\longleftarrow}} (\mathbf{A} \times [0])$$

in $\mathbb{T}$. We adopt the above writing of cosimplicial object and avoid using subscripts in order to differentiate it from the notation used in $\mathbf{A}_n(A, B)$. The functor T then maps the above cosimplicial object to generate a new cosimplicial object

$$\cdots \; T(\mathbf{A} \times [2]) \underset{T(id\times\delta_2^0)}{\overset{T(id\times\delta_2^2)}{\longleftarrow}} T(\mathbf{A} \times [1]) \underset{T(id\times\delta_1^0)}{\overset{T(id\times\delta_1^1)}{\longleftarrow}} T(\mathbf{A} \times [0]).$$

in $\mathbb{T}$. This enables one to define an induced simplicial object

$$\cdots \underset{d_2^0}{\overset{d_2^2}{\longrightarrow}} (T(\mathbf{A} \times [1]), \mathbf{Top}) \underset{d_1^0}{\overset{d_1^1}{\longrightarrow}} (T(\mathbf{A} \times [0]), \mathbf{Top}),$$

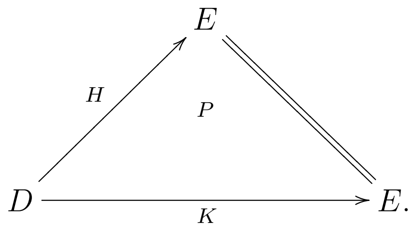
whose n-simplexes are all induced homotopy $(\mathbf{A} \times [n])$-diagrams, and the face d_n^i and the degeneracy s_n^i operators are defined by $d_n^i(D) = DT(id \times \delta_n^i)$ and $s_n^i(D) = DT(id \times \sigma_n^i)$, where D is a homotopy $(\mathbf{A} \times [n-1])$-diagram.

The two lowest dimensions of the above induced construction defines a morphism between a pair of homotopy $\mathbf{A}$-diagrams.

Definition 3.3.3 Let D and E be homotopy $\mathbf{A}$-diagrams. A homotopy homomorphism $H : D \longrightarrow E$ is defined to be a homotopy $(\mathbf{A} \times [1])$-diagram, $H : T(\mathbf{A} \times [1]) \longrightarrow \mathbf{Top}$, such that $d_1^0(H) = D$ and $d_1^1(H) = E$.

Further one additional dimension will then classify homotopic homomorphisms between homotopy $\mathbf{A}$-diagrams.

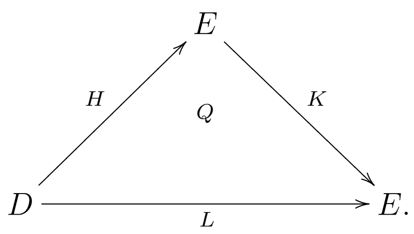
Definition 3.3.4 Two homotopy homomorphisms $H, K : D \longrightarrow E$ are called simplicial homotopic if there is a 2-simplex P, i.e. a homotopy $(\mathbf{A}\times[2])$-diagram, such that $d_2^0(P) = s_2^0(E)$, $d_2^1(P) = K$ and $d_2^2(P) = H$. Diagrammatically described by



We claim, cf. Vogt [55], or Cordier and Porter [13], [14] that the collection of homotopy **A**-diagrams and simplicial homotopy classes of homotopy homomorphisms forms a category. This will be denoted by **Coh(A,Top)** and will be discuss in detail generally later.

For a given two homotopy homomorphisms $H : D \longrightarrow E$ and $K : E \longrightarrow F$, there is a problem when one tries to define a composition $KH : D \longrightarrow F$. The idea to avoid this problem is by constructing another homotopy homomorphism $L : D \longrightarrow F$ as a composite of H and K.

Definition 3.3.5 Let $H : D \longrightarrow E$ and $K : E \longrightarrow F$ be homotopy homomorphisms. We call $L : D \longrightarrow F$ a composite of H and K if there is a 2-simplex Q, i.e. a homotopy $(\mathbf{A} \times [2])$-diagram, such that $d_2^0(Q) = K$, $d_2^1(Q) = L$ and $d_2^2(Q) = H$. Diagrammatically described by



3.4 $\mathcal{S}$-Enriched Homotopy Coherent Diagrams

As above, the category **Top** may be enriched to give an example of a (locally Kan) $\mathcal{S}$-category, denoted by $\mathbf{Top}_{\mathcal{S}}$, and $\mathbb{S}(\mathbf{A})$ is the DKC $\mathcal{S}$-category on **A** (see Subsections 2.4.2 and 2.4.4). Cordier [12] reformulated Boardman-Vogt's **Top**-enriched homotopy **A**-diagram to consider its simplicial analogue.

With a slight modification on the monoid multiplication, i.e. using $t_i * t_{i+1} = max(t_i, t_{i+1})$ instead of Boardman-Vogt's $t_i * t_{i+1} = t_i t_{i+1}$, Cordier identified Boardman-Vogt's homotopy **A**-diagram as a homotopy coherent diagram of type **A** in **Top**. We recall briefly his alternative description, compared to the earlier one of Boardman-Vogt's.

Definition 3.4.1 A homotopy coherent diagram of type **A** in **Top** is an assignment as follows: for each object A of **A** a space $F(A)$, to each pair of objects (A, B) of **A** a continuous map

$$F_{A,B} : T(\mathbf{A})(A, B) \longrightarrow \mathbf{Top}(FA, FB),$$

satisfying the following conditions
$F_{A,B}(f_0, t_1, f_1, ..., f_n, t_n)(x) =$

$$\begin{cases} F_{A,B}(f_1, t_2, f_2, ..., f_n, t_n)(x) & \mathit{if}\ f_0 = id, \\ F_{A,B}(f_0, t_1, f_1, ..., max t_{i+1}, t_i, ..., f_n, t_n)(x) & \mathit{if}\ f_i = id \\ F_{A,B}(f_0, t_1, f_1, ..., t_{n-1}, f_{n-1})(x) & \mathit{if}\ f_n = id, \\ F_{A,B}(f_0, t_1, f_1, ..., f_i f_{i-1}, ..., f_n, t_n)(x) & \mathit{if}\ t_i = 0, \\ F_{A_i,B}(f_i, t_{i+1}, f_{i+1}, ..., f_n, t_n)(F_{A,A_i}(f_0, t_1, f_1, ..., f_{i-1}, t_{i-1})(x)) & \mathit{if}\ t_i = 1, \end{cases}$$

where $A_i = \mathit{cod}(f_{i-1}) = \mathit{dom}(f_i)$.

Cordier then showed that Boardman-Vogt's **Top**-enriched homotopy coherent diagrams above are in fact equivalent to $\mathcal{S}$-enriched homotopy coherent diagrams, up to the replacement of the monoid structure on the interval by max. The above definition then can be shown to be equivalent to that of an $\mathcal{S}$-functor between the $\mathcal{S}$-categories, $\mathbb{S}(\mathbf{A})$ and $\mathbf{Top}_{\mathcal{S}}$. We shall redefine the notion of homotopy **A**-diagram using Cordier's equivalence.

Definition 3.4.2 A homotopy coherent diagram of type $\mathbf{A}$ in $\mathbf{Top}_{\mathcal{S}}$ is an $\mathcal{S}$-functor

$$F : \mathbb{S}(\mathbf{A}) \longrightarrow \mathbf{Top}_{\mathcal{S}}.$$

To look for an induced homotopy coherent diagram of type $\mathbf{A}$ in $\mathbf{Top}_{\mathcal{S}}$, it is reasonable first to note the following lemma.

Lemma 3.4.3 If $\theta : \mathbf{A} \longrightarrow \mathbf{B}$ is a functor, then $\mathbb{S}(\theta) : \mathbb{S}(\mathbf{A}) \longrightarrow \mathbb{S}(\mathbf{B})$ is an $\mathcal{S}$-functor.

Equivalent to Definition 3.2.3, we have the following concept.

Definition 3.4.4 If $\theta : \mathbf{A} \longrightarrow \mathbf{B}$ is a functor and G is a homotopy coherent diagram of type $\mathbf{B}$ in $\mathbf{Top}_{\mathcal{S}}$

$$G : \mathbb{S}(\mathbf{B}) \longrightarrow \mathbf{Top}_{\mathcal{S}},$$

then $G\mathbb{S}(\theta)$ is a homotopy coherent diagram of type $\mathbf{A}$ in $\mathbf{Top}_{\mathcal{S}}$

$$G\mathbb{S}(\theta) : \mathbb{S}(\mathbf{A}) \longrightarrow \mathbf{Top}_{\mathcal{S}}.$$

This idea will be used in the later construction of an induced simplicial object, whose n-simplexes are all induced homotopy coherent diagram of type $(\mathbf{A} \times [n])$ in $\mathbf{Top}_{\mathcal{S}}$.

3.4.1 Coherent Maps

Our intention here is to give the analogous definitions to those given in Subsection 3.3.1, and where necessary to replace terminology used by Boardman-Vogt with an equivalent one. We describe first some preliminary concepts which related to those definitions, particularly the cosimplicial object of the collection of DKC $\mathcal{S}$-categories $\mathbb{S}(\mathbf{A})$ on $\mathbf{A}$.

Suppose Δ is the category whose objects are the non-empty finite totally ordered sets $[n] = \{0, 1, ..., n\}$, and morphisms are increasing maps $\mu : [m] \longrightarrow [n]$ such that $i \leq j$ implies $\mu(i) \leq \mu(j)$. Let Cat be the category of categories and functors, and SCat be the category of $\mathcal{S}$-categories and $\mathcal{S}$-functors.

For any category $\mathbf{A}$, form a cosimplicial object

$$\cdots \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; (\mathbf{A}\times[2]) \;\underset{id\times\delta_2^0}{\overset{id\times\delta_2^2}{\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}}}\; (\mathbf{A}\times[1]) \;\underset{id\times\delta_1^0}{\overset{id\times\delta_1^1}{\leftleftarrows}}\; (\mathbf{A}\times[0])$$

in Cat. Using the theory of the DKC $\mathcal{S}$-category on $(\mathbf{A}\times[n])$, define then a cosimplicial object

$$\cdots \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \mathbb{S}(\mathbf{A}\times[2]) \;\underset{\mathbb{S}(id\times\delta_2^0)}{\overset{\mathbb{S}(id\times\delta_2^2)}{\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}}}\; \mathbb{S}(\mathbf{A}\times[1]) \;\underset{\mathbb{S}(id\times\delta_1^0)}{\overset{\mathbb{S}(id\times\delta_1^1)}{\leftleftarrows}}\; \mathbb{S}(\mathbf{A}\times[0])$$

in SCat, whose objects of n-simplexes is the DKC $\mathcal{S}$-categories $\mathbb{S}(\mathbf{A}\times[n])$, and the face and degeneracy operators are defined by $\mathbb{S}(id\times\delta_n^i)$ and $\mathbb{S}(id\times\sigma_n^i)$. This enable one to define an induced simplicial object

$$\cdots \;\underset{d_2^0}{\overset{d_2^2}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}}}\; (\mathbb{S}(\mathbf{A}\times[1]),\mathbf{Top}_{\mathcal{S}}) \;\underset{d_1^0}{\overset{d_1^1}{\rightrightarrows}}\; (\mathbb{S}(\mathbf{A}\times[0]),\mathbf{Top}_{\mathcal{S}}),$$

whose n-simplexes are all induced homotopy coherent diagrams of type $(\mathbf{A}\times[n])$ in $\mathbf{Top}_{\mathcal{S}}$, and the face d_n^i and the degeneracy s_n^i operators are defined by $d_n^i(F) = F\mathbb{S}(id\times\delta_n^i)$ and $s_n^i(F) = F\mathbb{S}(id\times\sigma_n^i)$, where F is a homotopy coherent diagram of type $(\mathbf{A}\times[n-1])$ in $\mathbf{Top}_{\mathcal{S}}$.

The two lowest dimensions will define a coherent map between a pair of homotopy coherent diagrams of type $\mathbf{A}$ in $\mathbf{Top}_{\mathcal{S}}$. We are replacing Boardman-Vogt's terminology of *homotopy homomorphism* with *coherent map.*

Definition 3.4.5 Let F and G be homotopy coherent diagrams of type $\mathbf{A}$ in $\mathbf{Top}_{\mathcal{S}}$. A coherent map $f : F \longrightarrow G$ is a homotopy coherent diagram of type $(\mathbf{A}\times[1])$ in $\mathbf{Top}_{\mathcal{S}}$,

$$f : \mathbb{S}(\mathbf{A}\times[1]) \longrightarrow \mathbf{Top}_{\mathcal{S}},$$

whose restriction to $\mathbf{A} \times \{(0)\}$ is F, and to $\mathbf{A} \times \{(1)\}$ is G.

Two coherent maps then are homotopic to each other if there exists a homotopy coherent diagram of type $(\mathbf{A} \times [2])$ in $\mathbf{Top}_{\mathcal{S}}$. We are again replacing Boardman-Vogt's terminology of *simplicial homotopy* with just *homotopy.*

Definition 3.4.6 Two coherent maps $f, g : F \longrightarrow G$ are homotopic if there is a 2-simplex α, i.e. a homotopy coherent diagram of type $(\mathbf{A} \times [2])$ in $\mathbf{Top}_{\mathcal{S}}$,

$$\alpha : \mathbb{S}(\mathbf{A} \times [2]) \longrightarrow \mathbf{Top}_{\mathcal{S}},$$

such that $d_2^0(\alpha) = s_2^0(F)$, $d_2^1(\alpha) = g$ and $d_2^2(\alpha) = f$.

Once more, equivalently, we can form the same category **Coh(A,Top$_{\mathcal{S}}$)** of homotopy coherent diagrams of type $\mathbf{A}$ in $\mathbf{Top}_{\mathcal{S}}$ and homotopy classes of coherent maps. We will examine the general case in Subsection 3.5.2.

We define a *composite* of a pair of coherent maps to be given by a homotopy coherent diagram of type $(\mathbf{A} \times [2])$ in $\mathbf{Top}_{\mathcal{S}}$.

Definition 3.4.7 Let $f : F \longrightarrow G$ and $g : G \longrightarrow H$ be coherent maps. We call $h : F \longrightarrow H$ a composite of f and g if there is a 2-simplex β, i.e. a homotopy coherent diagram of type $(\mathbf{A} \times [2])$ in $\mathbf{Top}_{\mathcal{S}}$,

$$\beta : \mathbb{S}(\mathbf{A} \times [2]) \longrightarrow \mathbf{Top}_{\mathcal{S}},$$

such that $d_2^0(\beta) = g$, $d_2^1(\beta) = h$ and $d_2^2(\beta) = f$.

3.5 Generalized $\mathcal{S}$-Enriched Homotopy Coherent Diagrams

The fact that $\mathbf{Top}_{\mathcal{S}}$ is an example of a (locally Kan) $\mathcal{S}$-category enables one to define a more general form of homotopy coherent diagram, i.e. of type $\mathbf{A}$ in $\mathbf{B}_{\mathcal{S}}$, an arbitrary (locally Kan) $\mathcal{S}$-category, cf. Cordier and Porter [13], [14].

Definition 3.5.1 Let $\mathbf{A}$ be any category and $\mathbf{B}_{\mathcal{S}}$ an $\mathcal{S}$-category. A homotopy coherent diagram of type $\mathbf{A}$ in $\mathbf{B}_{\mathcal{S}}$ is an $\mathcal{S}$-functor

$$F : \mathbb{S}(\mathbf{A}) \longrightarrow \mathbf{B}_{\mathcal{S}}.$$

The generalized version to Corollary 3.4.4 is in the following statement. It will be use later to construct an induced simplicial object whose n-simplexes are all induced homotopy coherent diagram of type $(\mathbf{A} \times [n])$ in $\mathbf{B}_{\mathcal{S}}$.

Corollary 3.5.2 If $\theta : \mathbf{C} \longrightarrow \mathbf{D}$ is a functor and $G : \mathbb{S}(\mathbf{D}) \longrightarrow \mathbf{B}_{\mathcal{S}}$ is a homotopy coherent diagram of type $\mathbf{D}$ in $\mathbf{B}_{\mathcal{S}}$, then $G\mathbb{S}(\theta) : \mathbb{S}(\mathbf{C}) \longrightarrow \mathbf{B}_{\mathcal{S}}$ is a homotopy coherent diagram of type $\mathbf{C}$ in $\mathbf{B}_{\mathcal{S}}$.

3.5.1 Coherent Maps

Here and in the next subsection we will give the generalized statements to all of the definitions and constructions found earlier in Subsection 3.3.1, or equivalently in Subsection 3.4.1. This is now routine but is included for completeness.

Consider first the similar construction as being discussed in Subsection 3.4.1 up to those of an induced simplicial object, i.e. an induced simplicial object

$$\cdots \underset{d_2^0}{\overset{d_2^2}{\Rrightarrow}} (\mathbb{S}(\mathbf{A} \times [1]), \mathbf{B}_{\mathcal{S}}) \underset{d_1^0}{\overset{d_1^1}{\rightrightarrows}} (\mathbb{S}(\mathbf{A} \times [0]), \mathbf{B}_{\mathcal{S}}),$$

whose n-simplexes are all induced homotopy coherent diagrams of type $(\mathbf{A} \times [n])$ in $\mathbf{B}_{\mathcal{S}}$, and the face d_n^i and the degeneracy s_n^i operators are defined by $d_n^i(F) = F\mathbb{S}(id \times \delta_n^i)$ and $s_n^i(F) = F\mathbb{S}(id \times \sigma_n^i)$, where F is a homotopy coherent diagram of type $(\mathbf{A} \times [n-1])$ in $\mathbf{B}_{\mathcal{S}}$.

Consequently, for any given two homotopy coherent diagrams of type $\mathbf{A}$ in $\mathbf{B}_{\mathcal{S}}$, a coherent map between them will be a homotopy coherent diagram of type $(\mathbf{A} \times [1])$ in $\mathbf{B}_{\mathcal{S}}$.

Definition 3.5.3 Let F and G are homotopy coherent diagrams of type $\mathbf{A}$ in $\mathbf{B}_{\mathcal{S}}$. A coherent map $f : F \longrightarrow G$ is a homotopy coherent diagram of type $(\mathbf{A} \times [1])$ in $\mathbf{B}_{\mathcal{S}}$,

$$f : \mathbb{S}(\mathbf{A} \times [1]) \longrightarrow \mathbf{B}_{\mathcal{S}},$$

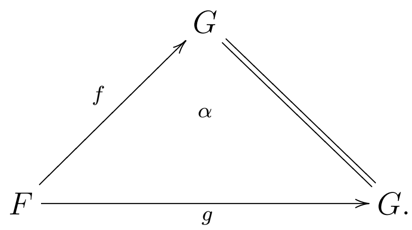
agreeing with F on $\mathbf{A} \times \{(0)\}$ and with G on $\mathbf{A} \times \{(1)\}$.

The general idea of a homotopy between a pair of coherent maps is in the following.

Definition 3.5.4 Two coherent maps $f, g : F \longrightarrow G$ are homotopic if there is a 2-simplex α, i.e. a coherent diagram of type $(\mathbf{A} \times [2])$ in $\mathbf{B}_{\mathcal{S}}$,

$$\alpha : \mathbb{S}(\mathbf{A} \times [2]) \longrightarrow \mathbf{B}_{\mathcal{S}},$$

such that $d_2^0(\alpha) = s_2^0(F)$, $d_2^1(\alpha) = g$ and $d_2^2(\alpha) = f$. These can diagrammatically described by

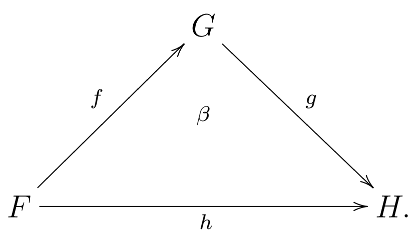


The composition of two coherent maps is given by a homotopy coherent diagram of type $(\mathbf{A} \times [2])$ in $\mathbf{B}_{\mathcal{S}}$.

Definition 3.5.5 Let $f : F \longrightarrow G$ and $g : G \longrightarrow H$ be coherent maps. We call $h : F \longrightarrow H$ a composite of f and g if there is a 2-simplex β, i.e. a homotopy coherent diagram of type $(\mathbf{A} \times [2])$ in $\mathbf{B}_{\mathcal{S}}$,

$$\beta : \mathbb{S}(\mathbf{A} \times [2]) \longrightarrow \mathbf{B}_{\mathcal{S}},$$

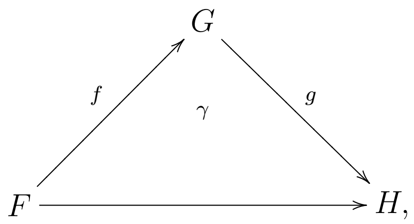
such that $d_2^0(\beta) = g$, $d_2^1(\beta) = h$ and $d_2^2(\beta) = f$. Diagrammatically described by



3.5.2 The Category Coh(A,B$_{\mathcal{S}}$)

We give here the general construction of the category mentioned in Subsections 3.3.1 and 3.4.1, i.e. the category **Coh(A,B$_{\mathcal{S}}$)**. The following passage explains the axioms of composition, associativity and identities for this category.

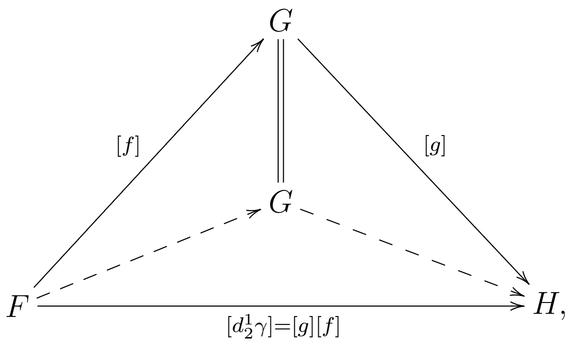
For $\mathbf{B}_{\mathcal{S}}$ locally Kan, observe that a diagram of coherent maps



will give a homotopy coherent diagram of type $(\mathbf{A} \times [2])$ in $\mathbf{B}_{\mathcal{S}}$,

$$\gamma : \mathbb{S}(\mathbf{A} \times [2]) \longrightarrow \mathbf{B}_{\mathcal{S}},$$

such that $d_2^0(\gamma) = g$ and $d_2^2(\gamma) = f$. We would like to take $d_2^1(\gamma)$ as the composite of f and g, however this would not be well defined, as there may be more than one extension possible. If on the other hand we pass to homotopy classes of coherent maps, one easily sees that the weak Kan condition on $\mathbf{B}_{\mathcal{S}}$ implies that $[d_2^1\gamma] = [g][f]$, diagrammatically described by

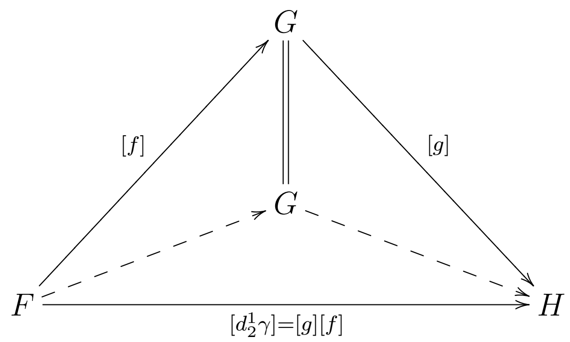


gives a well defined composition, as any two choices of filler differ by a homotopy.

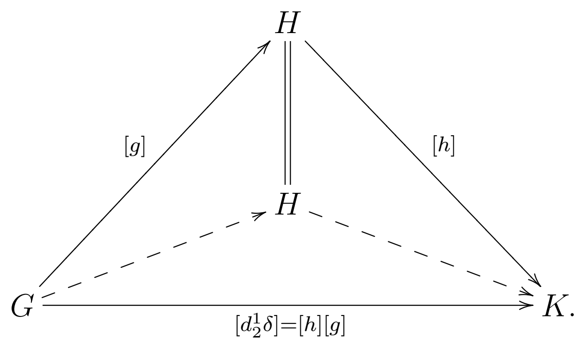
The above composite of homotopy classes of coherent maps does satisfy the axiom of associativity. (This is well known, but the proof are needed for completeness.)

Lemma 3.5.6 $[h]([g][f]) = ([h][g])[f]$.

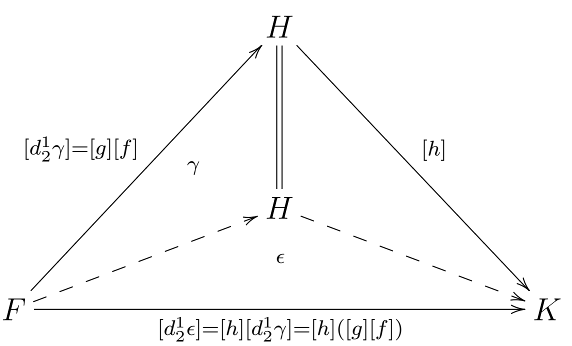
Proof: Let $[d_2^1\gamma] = [g][f]$ and $[d_2^1\delta] = [h][g]$, represented by



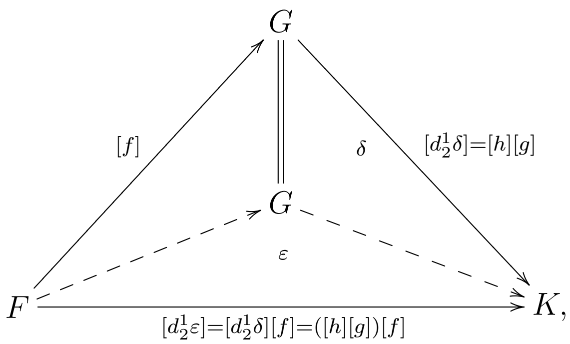
and



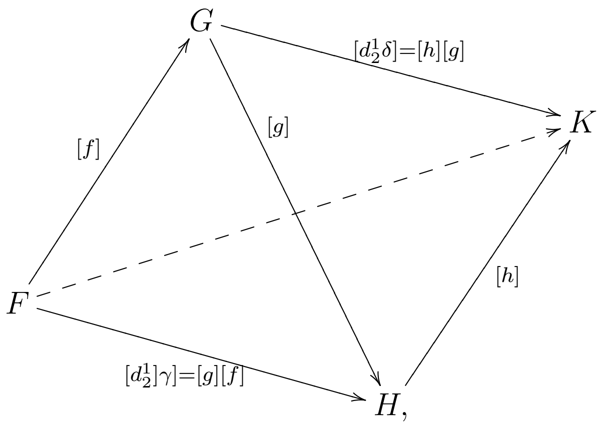
Analyse that a suitable composition of the above two diagrams will give the following two diagrams,



and



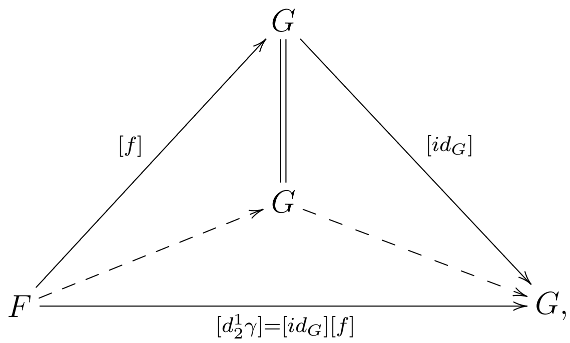
where ϵ and ε are the new homotopy coherent diagram of type $(\mathbf{A} \times [2])$ in $\mathbf{B}_{\mathcal{S}}$. Both diagrams can be combined to form the following diagram



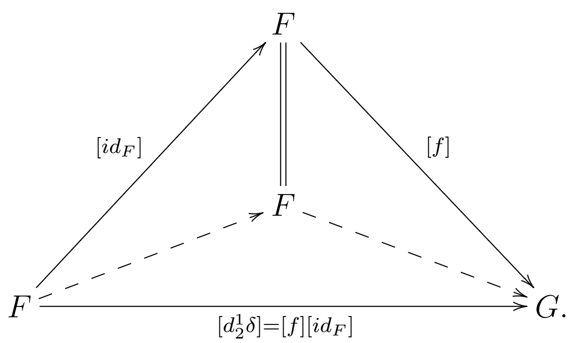
where the back $\{-- >\}$ arrow represent $[h]([g][f]) = ([h][g])[f]$. ■

The axiom of identities then is a consequences of the axiom of composition.

Lemma 3.5.7 For any homotopy class of coherent maps $[f] : F \longrightarrow G$, the left identity of $[f]$ is $[id_G]$ such that verification of axiom $[id_G][f] = [f]$ is represented by



and the right identity of $[f]$ is $[id_F]$ such that the axiom $[f][id_F] = [f]$ is represented by



The above axioms of composition, associativity and identities define the category **Coh(A,B$_\mathcal{S}$)** of homotopy coherent diagrams of type **A** in **B**$_\mathcal{S}$ and homotopy classes of coherent maps.

Chapter 4

Coverings and Homotopy Coherent Diagrams

4.1 Introduction

Our intention here is to study a specific type of naturally occuring homotopy coherent functor, that is, an $\mathcal{S}$-functor

$$C(X; -) : \mathbb{S}(\mathbf{Cov_{\leq}(X)}) \longrightarrow \mathcal{S}_{\mathcal{S}},$$

where $\mathbb{S}(\mathbf{Cov_{\leq}(X)})$ is the DKC $\mathcal{S}$-category on $\mathbf{Cov_{\leq}(X)}$, the "directed" category of open coverings of a space X, and $\mathcal{S}_{\mathcal{S}}$ is an $\mathcal{S}$-category on $\mathcal{S}$ enriched over itself.

In Section 4.2, we briefly recall the properties of $\mathbf{Cov_{\leq}(X)}$ and some material on Čech complexes. In Section 4.3, we note some other developments connected to Čech complex theory and prove the Classical Lemma on the Čech complex functor between $\mathbf{Cov_{\leq}(X)}$ and $\mathbf{Ho}(\mathcal{S})$.In Section 4.4, we lift the previous Čech complex functor to form the main result of this chapter, i.e. the *homotopy coherent Čech complex functor* between $\mathcal{S}$-categories $\mathbb{S}(\mathbf{Cov_{\leq}(X)})$ and $\mathcal{S}_{\mathcal{S}}$.

4.2 Coverings and Čech Complexes

We take most of the material in the following from Dowker [22], Spanier [49] and Porter [43], [44], [45], [46].

4.2.1 The Category $\mathbf{Cov_{\leq}(X)}$

We are mainly concerned here with the structure of the "directed" category $\mathbf{Cov_{\leq}(X)}$, whose objects are open coverings and whose only morphisms are given by order of open coverings.

Definition 4.2.1 Suppose X is a space and $\mathcal{U} = \{U_i | i \in I\}$ is an open covering of X. An open covering $\mathcal{U}$ is a refinement of $\mathcal{V}$ if for each $U \in \mathcal{U}$, there exist $V \in \mathcal{V}$ such that $U \subset V$. A refinement map is an assignment $\varphi : \mathcal{U} \longrightarrow \mathcal{V}$ such that for each $U \in \mathcal{U}$, $U \subset \varphi(U)$. Denote $\mathcal{U} \leq \mathcal{V}$ if there exists a refinement map $\varphi : \mathcal{U} \longrightarrow \mathcal{V}$.

The collection of open coverings and its order gives an ordered set.

Definition 4.2.2 The pair $(Cov(X), \leq)$ of set of open coverings and its order gives an ordered set satisfying:
(i) $\mathcal{U} \leq \mathcal{U}$ for all $\mathcal{U} \in Cov(X)$,
(ii) $\mathcal{U} \leq \mathcal{V}$ and $\mathcal{V} \leq \mathcal{W}$ implies $\mathcal{U} \leq \mathcal{W}$,
(iii) given $\mathcal{U}, \mathcal{V} \in Cov(X)$, there is some $\mathcal{W} \in Cov(X)$ with $\mathcal{W} \leq \mathcal{U}$ and $\mathcal{W} \leq \mathcal{V}$.

The ordered set, $Cov(X)$, then can be regarded as a "directed" category, $\mathbf{Cov_{\leq}(X)}$, with $Cov(X)$ as the set of objects and with just a single morphism $\mathcal{U} \leq \mathcal{V}$ if there is a refinement map $\varphi : \mathcal{U} \longrightarrow \mathcal{V}$.

4.2.2 Čech Complexes

For any open covering $\mathcal{U}$ of X, there is associated a simplicial set, denoted by $C(X; \mathcal{U})$, which will be called the Čech complex, or nerve, of $\mathcal{U}$. Simplices in it will be defined by non-empty intersection of finite open sets in $\mathcal{U}$.

Definition 4.2.3 Suppose $\mathcal{U}$ is an open covering of X. The Čech complex of $\mathcal{U}$ is a simplicial set $C(X;\mathcal{U}) : \Delta^{op} \longrightarrow$ **Sets** with typical n-simplex, a $(n+1)$-tuple $< U_0, ..., U_n >$ with $U_i \in \mathcal{U}$ such that $\bigcap_{i=0}^{n} U_i \neq \emptyset$. The face and degeneracy operators are

$$\begin{aligned} d_i(< U_0, ..., U_{i-1}, U_i, U_{i+1}, ..., U_n >) &= < U_0, ..., U_{i-1}, U_{i+1}, ..., U_n >, \\ s_i(< U_0, ..., U_{i-1}, U_i, U_{i+1}, ..., U_n >) &= < U_0, ..., U_{i-1}, U_i, U_i, U_{i+1}, ..., U_n > . \end{aligned}$$

If $\mathcal{U} \leq \mathcal{V}$, given by a refinement map $\varphi : \mathcal{U} \longrightarrow \mathcal{V}$, then a map of Čech complexes is given by

$$C(X;\varphi) : C(X;\mathcal{U}) \longrightarrow C(X;\mathcal{V}),$$
$$< U_0, ..., U_n > \mapsto < \varphi(U_0), ..., \varphi(U_n) >.$$

This map is hopelessly ill defined, as it depends explicitly on φ not just on the order $\mathcal{U} \leq \mathcal{V}$, but it is unique up to homotopy and defines a functor, as explained in Subsection 4.3.2 below.

4.3 The Classical Lemma

Here we discuss some other approaches in studying ideas related to the Čech complex functor. Next, for some obvious reasons, we pick the Čech complex functor approach and analyse all the classical properties necessary for our main theorem in Section 4.4.

4.3.1 Remarks

Besides the Čech complex functor, which will be discussed in the next subsection as the turning point in the development of the main theorem in this thesis, there are other similar functors, i.e. the Vietoris complex functor, cf. Dowker [21], [22] and Porter [43], [46], and the Thomas-Abels-Holz complex functor, taken from Abels and Holz [1] or, much older, from Thomas [53].

Suppose $\mathbf{Cov}_{\leq}\mathbf{(X)}$ is the "directed" category of open coverings and the order as above. Let $\mathcal{U}$ be an open covering of X. The *Vietoris complex of*

$\mathcal{U}$, denoted by $V(X;\mathcal{U})$, is a simplicial set with typical n-simplex an $(n+1)$-tuple $< x_0, ..., x_n >$ of points of X such that there is an open set U in $\mathcal{U}$ with $x_i \in U$ for $i = 0, 1, ..., n$. If $\mathcal{U} \leq \mathcal{V}$ then there exists a *map of Vietoris complexes* given by

$$V(X;\varphi) : V(X;\mathcal{U}) \longrightarrow V(X : \mathcal{V})$$
$$< x_0, ..., x_n >_{\mathcal{U}} \mapsto < x_0, ..., x_n >_{\mathcal{V}}.$$

The resulted map $V(X; -) : \mathbf{Cov}_{\leq}\mathbf{(X)} \longrightarrow \mathcal{S}$ is a functor, called the *Vietoris complex functor*. We note $V(X;\varphi)$ is independent of φ.

For any open covering $\mathcal{U}$ of X, the *Thomas-Abels-Holz complex of* $\mathcal{U}$, denoted by $T(X;\mathcal{U})$, is the simplicial set with typical n-simplex a $(n+1)$-tuple $< (x_0, U_0), ..., (x_n, U_n) >$ such that $x_i \in \bigcap_{i=0}^{n} U_i$. For $\mathcal{U} \leq \mathcal{V}$, given by $\varphi : \mathcal{U} \longrightarrow \mathcal{V}$, there exists a *map of Thomas-Abels-Holz complexes* given by

$$T(X;\varphi) : T(X;\mathcal{U}) \longrightarrow T(X;\mathcal{V})$$
$$< (x_0, U_0), ..., (x_n, U_n) > \mapsto < (x_0, \varphi(U_0)), ..., (x_n, \varphi(U_n)) >.$$

This map also is hopelessly dependent on φ, but it is unique up to homotopy and will give the *Thomas-Abels-Holz complex functor*

$$T(X; -) : \mathbf{Cov}_{\leq}\mathbf{(X)} \longrightarrow \mathbf{Ho}(\mathcal{S}).$$

It is interesting to note that the corresponding sheaf version of the Thomas-Abels-Holz complex will emerged later in the construction of "inclusion" étale Čech complexes as simplicial sheaves in Subsection 5.7.1.

4.3.2 The Čech Complex Functor

Here, we reprove the Classical Lemma on the *Čech complex functor*

$$C(X; -) : \mathbf{Cov}_{\leq}\mathbf{(X)} \longrightarrow \mathbf{Ho}(\mathcal{S}),$$

where $\mathbf{Ho}(\mathcal{S})$ is the category of simplicial sets and homotopy classes of simplicial maps.

Lemma 4.3.1 If $\varphi, \psi : \mathcal{U} \longrightarrow \mathcal{V}$ are refinement maps, then there is a homotopy

$$h : C(X;\mathcal{U}) \times \Delta[1] \longrightarrow C(X;\mathcal{V})$$

such that $C(X;\varphi) \simeq C(X;\psi)$.

Proof: Suppose $\varphi, \psi : \mathcal{U} \longrightarrow \mathcal{V}$ are refinement maps. These will induce two maps of Čech complexes

$$C(X;\varphi), C(X;\psi) : C(X;\mathcal{U}) \longrightarrow C(X;\mathcal{V}).$$

Suppose given $\sigma : \Delta[n] \longrightarrow C(X;\mathcal{U})$ that picks out $< U_0, ..., U_n >$. The resulting compositions

$$C(X;\varphi)\sigma, C(X;\psi)\sigma : \Delta[n] \longrightarrow C(X;\mathcal{V})$$

enable one to define a simplicial map

$$h_\sigma : \Delta[n] \times \Delta[1] \longrightarrow C(X;\mathcal{V})$$

such that

$$h_\sigma((0, ..., i, ..., n), (0, ..., 0, ..., 0)) =< \varphi(U_0), ..., \varphi(U_i), ..., \varphi(U_n) >$$

and

$$h_\sigma((0, ..., i, ..., n), (1, ..., 1, ..., 1)) =< \psi(U_0), ..., \psi(U_i), ..., \psi(U_n) >,$$

tare compatible with the faces and degeneracies of σ.

An additional step now is to write both conditions of h_σ in one formula. This will be technically useful for controlling information when dealing with higher dimensional (coherence) homotopies later. It can be done by analysing first the nature of a typical $(n+1)$-simplex of $\Delta[n]\times\Delta[1]$. Generally, a typical $(n+1)$-simplex of $\Delta[n] \times \Delta[1]$ is represented by

$$((0, ..., i, i, ..., n), (0, ..., 0_i, 1_i, ..., 1)),$$

that is, when an element of $\Delta[n]$ is repeated, an element of $\Delta[1]$ increases from 0 to 1. For the reason that elements of $\Delta[n]$ determine elements of $C(X;\mathcal{U})$, a typical $(n+1)$-simplex of $C(X;\mathcal{U}) \times \Delta[1]$ becomes

$$(< U_0, ..., U_i, U_i, ..., U_n >, (0, ..., 0_i, 1_i, ..., 1)).$$

This then maps by $C(X;\varphi)$ and $C(X;\psi)$ into $C(X;\mathcal{V})$ to form

$$(< \varphi(U_0), ..., \varphi(U_i), \psi(U_i), ..., \psi(U_n) >.$$

So, our original situations for h_σ now becomes

$$h_\sigma((0, ..., i, i, ..., n), (0, ..., 0_i, 1_i, ..., 1)) =< \varphi(U_0), ..., \varphi(U_i), \psi(U_i), ..., \psi(U_n) >$$

and this gives a homotopy

$$h : C(X;\varphi) \simeq C(X;\psi) : C(X;\mathcal{U}) \times \Delta[1] \longrightarrow C(X;\mathcal{V})$$

where

$$\begin{array}{c} h(< U_0, ..., U_i, U_i, ..., U_n >, (0, ..., 0_i, 1_i, ..., 1)) = \\ < \varphi(U_0), ..., \varphi(U_i), \psi(U_i), ..., \psi(U_n) >. \end{array}$$

■

We will show that the Classical Lemma can be lifted to an $\mathcal{S}$-functor using the augmentation functor

$$aug : \mathbb{S}(\mathbf{Cov_{\leq}(X)}) \longrightarrow \mathbf{Cov_{\leq}(X)}$$

and the canonical $\mathcal{S}$-functor

$$\pi_0 : \mathcal{S}_\mathcal{S} \longrightarrow \mathbf{Ho}(\mathcal{S}).$$

By combining all of the related ideas to form a diagram

$$\begin{array}{ccc} \mathbb{S}(\mathbf{Cov_{\leq}(X)}) & \xrightarrow{C(X;-)} & \mathcal{S}_\mathcal{S} \\ {\scriptstyle aug}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ \mathbf{Cov_{\leq}(X)} & \xrightarrow[C(X;-)]{} & \mathbf{Ho}(\mathcal{S}), \end{array}$$

then there is a lift of the "bottom" Čech complex functor to the "top" homotopy coherent Čech complex functor.

4.4 Homotopy Coherent Čech Complex Functors

The lift of the Čech complex functor

$$C(X;-) : \mathbf{Cov}_{\leq}(\mathbf{X}) \longrightarrow \mathbf{Ho}(\mathcal{S})$$

to the homotopy coherent Čech complex functor

$$C(X;-) : \mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X})) \longrightarrow \mathcal{S}_{\mathcal{S}},$$

will be proved in the following result.

Theorem 4.4.1 For any choice of refinement maps φ, there is a lift of the resulting data in Lemma 4.3.1 to an $\mathcal{S}$-functor

$$C(X;-) : \mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X})) \longrightarrow \mathcal{S}_{\mathcal{S}}.$$

We postpone the proof until later. Our intention in the following passage is to sketch the lowest dimensional cases for $(m-1)$-simplex and its $(m-1)$-order homotopies using the construction explained in Subsection 2.4.3 for the $\mathcal{S}$-category $\mathcal{S}_{\mathcal{S}}$ and in Subsection 2.4.4 for the DKC $\mathcal{S}$-category $\mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X}))$. These will needed for basic preparation of the general proof of Theorem 4.4.1.

1-Simplex and 1^{st}-order homotopy

By applying Lemma 4.3.1, let $\mathcal{U}_0,\mathcal{U}_1,\mathcal{U}_2$ objects in $\mathbf{Cov}_{\leq}(\mathbf{X})$ such that $\mathcal{U}_0 \leq \mathcal{U}_1 \leq \mathcal{U}_2$. In other words, there exist refinement maps $\varphi_{01} : \mathcal{U}_0 \longrightarrow \mathcal{U}_1$, $\varphi_{02} : \mathcal{U}_0 \longrightarrow \mathcal{U}_2$, $\varphi_{12} : \mathcal{U}_1 \longrightarrow \mathcal{U}_2$ and $\varphi_{12}\varphi_{01} : \mathcal{U}_0 \longrightarrow \mathcal{U}_2$ with the properties that, for all $U_i \in \mathcal{U}_0$, we have $U_i \subset \varphi_{01}(U_i)$, $U_i \subset \varphi_{02}(U_i)$ and $U_i \subset \varphi_{12}\varphi_{01}(U_i)$, and, for all $V_j \in \mathcal{U}_1$, we have $V_j \subset \varphi_{12}(V_j)$. However φ_{02} need not be equal to $\varphi_{12}\varphi_{01}$. Note that φ_{ij} are not maps in $\mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X}))$ but are choices of representatives for $\mathcal{U}_i \leq \mathcal{U}_j$.

Parallel to the ideas put in example $\mathbb{S}[2](0,2) \cong \Delta[1]$, we have $\mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X}))(\mathcal{U}_0,\mathcal{U}_2)_0$ contains a 0-simplex $\{(\varphi_{02});(\varphi_{01},\varphi_{12})\}$ and $\mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X}))(\mathcal{U}_0,\mathcal{U}_2)_1$ contains a 1-simplex $\{((\varphi_{02}));((\varphi_{01}),(\varphi_{12}));((\varphi_{01},\varphi_{12}))\}$ such that $d_0((\varphi_{01},\varphi_{12})) = (\varphi_{01},\varphi_{12})$ and $d_1((\varphi_{01},\varphi_{12})) = (\varphi_{02})$. These 0- and 1-simplex can be presented by a diagram

$$(\varphi_{02}) \xrightarrow{((\varphi_{01},\varphi_{12}))} (\varphi_{01},\varphi_{12})$$

in $\mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{U}_0,\mathcal{U}_2)$.

The two vertices of the diagram are associated to two maps of Čech complexes,

$$\begin{gathered} C\Big(X;(\varphi_{02})\Big) : C(X;\mathcal{U}_0) \longrightarrow C(X;\mathcal{U}_2) \\ < U_0, ..., U_n > \mapsto < \varphi_{02}(U_0), ..., \varphi_{02}(U_n) > \end{gathered}$$

and

$$\begin{gathered} C\Big(X;(\varphi_{01},\varphi_{12})\Big) : C(X;\mathcal{U}_0) \longrightarrow C(X;\mathcal{U}_2) \\ < U_0, ..., U_n > \mapsto < \varphi_{12}\varphi_{01}(U_0), ..., \varphi_{12}\varphi_{01}(U_n) > . \end{gathered}$$

The edge of the diagram then is associated to the 1^{st}-order homotopy

$$C\Big(X;((\varphi_{01},\varphi_{12}))\Big) : C(X;\mathcal{U}_0) \times \Delta[1] \longrightarrow C(X;\mathcal{U}_2)$$

such that

$$\begin{gathered} C\Big(X;((\varphi_{01},\varphi_{12}))\Big)(< U_0, ..., U_i, U_i, ..., U_n >, (0, ..., 0_i, 1_i, ..., 1)) = \\ < \varphi_{02}(U_0), ..., \varphi_{02}(U_i), \varphi_{12}\varphi_{01}(U_i), ..., \varphi_{12}\varphi_{01}(U_n) >. \end{gathered}$$

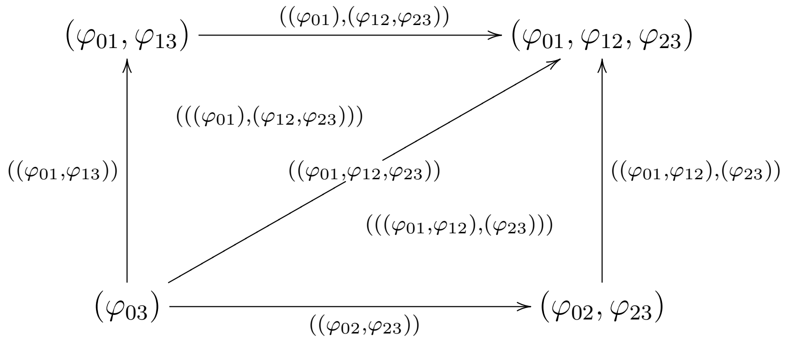
Briefly, there is a simplicial partial map ("partial" because there are infinitely many covers and refinement maps between $\mathcal{U}_0$ and $\mathcal{U}_2$, and we only considers $\mathcal{U}_1$)

$$C(X;-)_{\mathcal{U}_0,\mathcal{U}_2} : \mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{U}_0,\mathcal{U}_2) \longrightarrow \mathcal{S}_{\mathcal{S}}(C(X;\mathcal{U}_0), C(X;\mathcal{U}_2))$$

which map the 1-simplex $((\varphi_{01},\varphi_{12}))$ to 1^{st}-order homotopy $C\Big(X;((\varphi_{01},\varphi_{12}))\Big)$.

2-Simplexes and 2^{nd}-order homotopies

For the next higher dimension, let $\mathcal{U}_0,\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3$ be objects in $\mathbf{Cov_{\leq}(X)}$ such that $\mathcal{U}_0 \leq \mathcal{U}_1 \leq \mathcal{U}_2 \leq \mathcal{U}_3$ with the usual refinement maps and properties similar to the previous case. Using the similar parallel ideas in example $\mathbb{S}[3](0,3) \cong \Delta[1]^2$, the resulting square diagram of 0-, 1- and 2-simplexes can be presented by



in $\mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{U}_0, \mathcal{U}_3)$.

One then gets five 1^{st}-order homotopies:
(1) $C\Big(X;((\varphi_{01},\varphi_{13}))\Big) : C(X;\mathcal{U}_0) \times \Delta[1] \longrightarrow C(X;\mathcal{U}_3)$,
(2) $C\Big(X;((\varphi_{02},\varphi_{23}))\Big) : C(X;\mathcal{U}_0) \times \Delta[1] \longrightarrow C(X;\mathcal{U}_3)$,
(3) $C\Big(X;((\varphi_{01}),(\varphi_{12},\varphi_{23}))\Big) : C(X;\mathcal{U}_0) \times \Delta[1] \longrightarrow C(X;\mathcal{U}_3)$,
(4) $C\Big(X;((\varphi_{01},\varphi_{12}),(\varphi_{23}))\Big) : C(X;\mathcal{U}_0) \times \Delta[1] \longrightarrow C(X;\mathcal{U}_3)$,
(5) $C\Big(X;((\varphi_{01},\varphi_{12},\varphi_{23}))\Big) : C(X;\mathcal{U}_0) \times \Delta[1] \longrightarrow C(X;\mathcal{U}_3)$.

Such an example is the 5^{th} 1^{st}-order homotopy of the above list defined by

$$C\Big(X;((\varphi_{01},\varphi_{12},\varphi_{23}))\Big)(< U_0, ..., U_i, U_i, ..., U_n >, (0, ..., 0_i, 1_i, ..., 1)) = \\ < \varphi_{03}(U_0), ..., \varphi_{03}(U_i), \varphi_{23}\varphi_{12}\varphi_{01}(U_i), ..., \varphi_{23}\varphi_{12}\varphi_{01}(U_n) >.$$

They are all linked by two 2^{nd}-order homotopies. The first one is given by

$$C\Big(X;(((\varphi_{01}),(\varphi_{12},\varphi_{23})))\Big) : C(X;\mathcal{U}_0) \times \Delta[1]^2 \longrightarrow C(X;\mathcal{U}_3)$$

such that

$$C\Big(X;(((\varphi_{01}),(\varphi_{12},\varphi_{23})))\Big) \\ (< U_0, ..., U_i, U_i, ..., U_j, U_j, ..., U_n >, (0, ..., 0_i, 1_i, ..., 1_j, 2_j, ..., 2)) = \\ < \varphi_{03}(U_0), ..., \varphi_{03}(U_i), \varphi_{13}\varphi_{01}(U_i), ..., \varphi_{13}\varphi_{01}(U_j), \varphi_{23}\varphi_{12}\varphi_{01}(U_j), ..., \\ \varphi_{23}\varphi_{12}\varphi_{01}(U_n) > .$$

Observe that the definition of the above 2^{nd}-order homotopy follows a certain (maximal) path, given by

$$(\varphi_{03}) \longrightarrow (\varphi_{01}, \varphi_{13}) \longrightarrow (\varphi_{01}, \varphi_{12}, \varphi_{23}),$$

where the arrow could be read as "is replaced by". This also give a clue that it corresponds to the "upper" triangle of the square diagram and distinguishs it from the other one.

The second 2^{nd}-order homotopy is given by

$$C\Big(X; (((\varphi_{01}, \varphi_{12}), (\varphi_{23})))\Big) : C(X;\mathcal{U}_0) \times \Delta[1]^2 \longrightarrow C(X;\mathcal{U}_3)$$

such that

$$\begin{gathered} C\Big(X; (((\varphi_{01}, \varphi_{12}), (\varphi_{23})))\Big) \\ (< U_0, ..., U_i, U_i, ..., U_j, U_j, ..., U_n >, (0, ..., 0_i, 1_i, ..., 1_j, 2_j, ..., 2)) = \\ < \varphi_{03}(U_0), ..., \varphi_{03}(U_i), \varphi_{23}\varphi_{02}(U_i), ..., \varphi_{23}\varphi_{02}(U_j), \varphi_{23}\varphi_{12}\varphi_{01}(U_j), ..., \\ \varphi_{23}\varphi_{12}\varphi_{01}(U_n) > . \end{gathered}$$

The maximal path it follows is

$$(\varphi_{03}) \longrightarrow (\varphi_{02}, \varphi_{23}) \longrightarrow (\varphi_{01}, \varphi_{12}, \varphi_{23}).$$

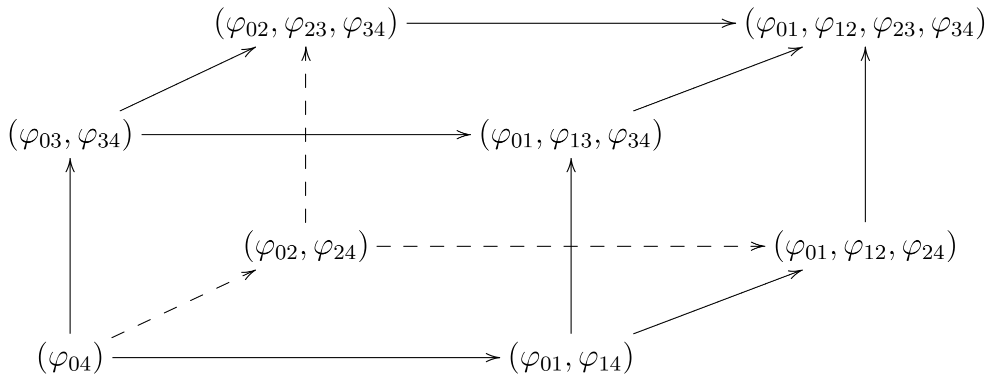
In brief, there is a simplicial partial map

$$C(X; -)_{\mathcal{U}_0,\mathcal{U}_3} : \mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{U}_0, \mathcal{U}_3) \longrightarrow \mathcal{S}_{\mathcal{S}}(C(X;\mathcal{U}_0), C(X;\mathcal{U}_3))$$

which maps 2-simplexes $(((\varphi_{01}), (\varphi_{12}, \varphi_{23})))$ and $(((\varphi_{01}, \varphi_{12}), (\varphi_{23})))$ to 2^{nd}-order homotopies $C\Big(X; (((\varphi_{01}), (\varphi_{12}, \varphi_{23})))\Big)$ and $C\Big(X; (((\varphi_{01}, \varphi_{12}), (\varphi_{23})))\Big)$.

3-Simplexes and 3^{rd}-order homotopies

The third dimensional case, being based on the cube face-diagram in example $\mathbb{S}[4](0, 4) \cong \Delta[1]^3$, will give us the following cube face-diagram



in $\mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X}))(\mathcal{U}_0, \mathcal{U}_4)$ with six 3-simplexes glued together:

(1) $((((\varphi_{01})), ((\varphi_{12}), (\varphi_{23}, \varphi_{34})))),$
(2) $((((\varphi_{01})), ((\varphi_{12}, \varphi_{23}), (\varphi_{34})))),$
(3) $((((\varphi_{01}), (\varphi_{12}, \varphi_{23})), ((\varphi_{34})))),$
(4) $((((\varphi_{01}, \varphi_{12}), (\varphi_{23})), ((\varphi_{34})))),$
(5) $((((\varphi_{01}, \varphi_{12})), ((\varphi_{23}), (\varphi_{34})))),$
(6) $((((\varphi_{01}), (\varphi_{12})), ((\varphi_{23}, \varphi_{34})))).$

These six 3-simplexes will gives us six 3^{rd}-order homotopies. The first one is defined as

$$C\Big(X; ((((\varphi_{01})), ((\varphi_{12}), (\varphi_{23}, \varphi_{34}))))\Big) : C(X; \mathcal{U}_0) \times \Delta[1]^3 \longrightarrow C(X; \mathcal{U}_4)$$

such that

$$\begin{gathered} C\Big(X; ((((\varphi_{01})), ((\varphi_{12}), (\varphi_{23}, \varphi_{34}))))\Big) \\ (< U_0, ..., U_i, U_i, ..., U_j, U_j, ..., U_k, U_k, ..., U_n >, \\ (0, ..., 0_i, 1_i, ..., 1_j, 2_j, ..., 2_k, 3_k, ..., 3)) = \\ < \varphi_{04}(U_0), ..., \varphi_{04}(U_i), \varphi_{14}\varphi_{01}(U_i), ..., \varphi_{14}\varphi_{01}(U_j), \varphi_{24}\varphi_{12}\varphi_{01}(U_j), ..., \\ \varphi_{24}\varphi_{12}\varphi_{01}(U_k), \varphi_{34}\varphi_{23}\varphi_{12}\varphi_{01}(U_k), ..., \varphi_{34}\varphi_{23}\varphi_{12}\varphi_{01}(U_n) >, \end{gathered}$$

which follows the maximal path

$$(\varphi_{04}) \longrightarrow (\varphi_{01}, \varphi_{14}) \longrightarrow (\varphi_{01}, \varphi_{12}, \varphi_{24}) \longrightarrow (\varphi_{01}, \varphi_{12}, \varphi_{23}, \varphi_{34}).$$

For simplicity, the following list below will present these corresponding 3^{rd}-order homotopies together with the maximal paths followed.

$$(2)\ C\Big(X;((((\varphi_{01})),((\varphi_{12},\varphi_{23}),(\varphi_{34}))))\Big):C(X;\mathcal{U}_0)\times\Delta[1]^3\longrightarrow C(X;\mathcal{U}_4),$$
$$(\varphi_{04})\longrightarrow(\varphi_{01},\varphi_{14})\longrightarrow(\varphi_{01},\varphi_{13},\varphi_{34})\longrightarrow(\varphi_{01},\varphi_{12},\varphi_{23},\varphi_{34}),$$

$$(3)\ C\Big(X;((((\varphi_{01}),(\varphi_{12},\varphi_{23})),((\varphi_{34}))))\Big):C(X;\mathcal{U}_0)\times\Delta[1]^3\longrightarrow C(X;\mathcal{U}_4),$$
$$(\varphi_{04})\longrightarrow(\varphi_{03},\varphi_{34})\longrightarrow(\varphi_{01},\varphi_{13},\varphi_{34})\longrightarrow(\varphi_{01},\varphi_{12},\varphi_{23},\varphi_{34}),$$

$$(4)\ C\Big(X;((((\varphi_{01},\varphi_{12}),(\varphi_{23})),((\varphi_{34}))))\Big):C(X;\mathcal{U}_0)\times\Delta[1]^3\longrightarrow C(X;\mathcal{U}_4),$$
$$(\varphi_{04})\longrightarrow(\varphi_{03},\varphi_{34})\longrightarrow(\varphi_{02},\varphi_{23},\varphi_{34})\longrightarrow(\varphi_{01},\varphi_{12},\varphi_{23},\varphi_{34}),$$

$$(5)\ C\Big(X;((((\varphi_{01},\varphi_{12})),((\varphi_{23}),(\varphi_{34}))))\Big):C(X;\mathcal{U}_0)\times\Delta[1]^3\longrightarrow C(X;\mathcal{U}_4),$$
$$(\varphi_{04})\longrightarrow(\varphi_{02},\varphi_{24})\longrightarrow(\varphi_{01},\varphi_{12},\varphi_{24})\longrightarrow(\varphi_{01},\varphi_{12},\varphi_{23},\varphi_{34}),$$

$$(6)\ C\Big(X;((((\varphi_{01}),(\varphi_{12})),((\varphi_{23},\varphi_{34}))))\Big):C(X;\mathcal{U}_0)\times\Delta[1]^3\longrightarrow C(X;\mathcal{U}_4)$$
$$(\varphi_{04})\longrightarrow(\varphi_{02},\varphi_{24})\longrightarrow(\varphi_{02},\varphi_{23},\varphi_{34})\longrightarrow(\varphi_{01},\varphi_{12},\varphi_{23},\varphi_{34}).$$

This actually defines a simplicial partial map

$$C(X;-)_{\mathcal{U}_0,\mathcal{U}_4}:\mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{U}_0,\mathcal{U}_4)\longrightarrow\mathcal{S}_{\mathcal{S}}(C(X;\mathcal{U}_0),C(X;\mathcal{U}_4))$$

which maps all of the six 3-simplexes to the corresponding six 3^{rd}-order homotopies. Compatibility of the various pieces is easily checked by inspection.

These three illustrations above help one to understand the lowest dimensional constructions for part of the proof of Theorem 4.4.1, i.e. the construction related to the formation of a simplicial partial map

$$C(X;-)_{\mathcal{U}_0,\mathcal{U}_m}:\mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{U}_0,\mathcal{U}_m)\longrightarrow\mathcal{S}_{\mathcal{S}}(C(X;\mathcal{U}_0),C(X;\mathcal{U}_m)),$$

which maps each of the $(m-1)$-simplexes to the corresponding $(m-1)$-order homotopies. Here we give the general proof of Theorem 4.4.1.

Proof of Theorem 4.4.1: We have to define a simplicially enriched functor

$$C(X;-):\mathbb{S}(\mathbf{Cov_{\leq}(X)})\longrightarrow\mathcal{S}_{\mathcal{S}}.$$

So given $\mathcal{U},\mathcal{V}\in\mathbf{Cov_{\leq}(X)}$, we have to define a simplicial map

$$C(X;-)_{\mathcal{U},\mathcal{V}} : \mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X}))(\mathcal{U},\mathcal{V}) \longrightarrow \mathcal{S}_{S}(C(X;\mathcal{U}),C(X;\mathcal{V})).$$

Thus if σ is an n-simplex of $\mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X}))(\mathcal{U},\mathcal{V})$, we have to define its image under $C(X;-)_{\mathcal{U},\mathcal{V}}$. The n-simplex σ corresponds to a string $(\varphi_{01},...,\varphi_{m-1m})$ of refinement maps (with $m \geq n$) together with a $(n+1)$-fold bracketting of that string. Here we have covers $\mathcal{U}_0 = \mathcal{U}, \mathcal{U}_1, ..., \mathcal{U}_m = \mathcal{V}$ and as always $\varphi_{kk+1} : \mathcal{U}_k \longrightarrow \mathcal{U}_{k+1}$. Thus to define $C(X;-)_{\mathcal{U},\mathcal{V}}$ on σ it suffices to define it relative to the sequence of covers $\mathcal{U}_0, \mathcal{U}_1, ..., \mathcal{U}_m$ that σ determines. (Call this sequence the carrier of σ, denoted by $\widetilde{\sigma}$). The n-simplex σ makes up part of the n-skeleton of the $(m+1)$-cube determined by $\mathcal{U}_0, \mathcal{U}_1, ..., \mathcal{U}_m$ (remembering that $\mathcal{S}[m](0,m) \cong \Delta[1]^{m-1}$), so bracketting correspond precisely to simplices of $\Delta[1]^{m-1}$. We therefore define $C(X;-)_{\mathcal{U}_0,\mathcal{U}_m}$ on that $(m-1)$-cube and hence define it on σ at the same time as on a lot of other simplices, "carried" by the same sequence of covers.

So let $\mathcal{U}_0, \mathcal{U}_1, ..., \mathcal{U}_m$ be objects in $\mathbf{Cov}_{\leq}(\mathbf{X})$ such that $\mathcal{U}_0 \leq \mathcal{U}_1 \leq ... \leq \mathcal{U}_m$ with the usual refinement maps and properties similar to those in the lowest dimensions. Parallel to the ideas described in Subsection 2.4.4. i.e. $\mathbb{S}[n](0,n) \cong \Delta[1]^{n-1}$, we will have a $(m-1)$-cube face-diagram in $\mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X}))(\mathcal{U}_0,\mathcal{U}_m)$.

We can inductively construct all the $(m-1)$-simplexes together with their bracketted structure based on the procedures laid for the lowest dimensions, but it is enough for us to give the general construction which presents all of these $(m-1)$-simplexes at once. For that purpose, we analyse back briefly all the previous three lowest dimensions in order to derive to the correct presentation for the general case. For the one 1-simplex within $\mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X}))(\mathcal{U}_0,\mathcal{U}_2)$, the associated maximal path

$$(\varphi_{02}) \longrightarrow (\varphi_{01},\varphi_{12})$$

will only give one possibility of breaking the refinement map φ_{02}, i.e. when $0 < \tau_1^1 = 1 < 2$.

For the two 2-simplexes within $\mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X}))(\mathcal{U}_0,\mathcal{U}_3)$, the associated maximal paths are:

$$(\varphi_{03}) \longrightarrow (\varphi_{01}, \varphi_{13}) \longrightarrow (\varphi_{01}, \varphi_{12}, \varphi_{23})$$

with $0 < \tau_1^2 = 1 < \tau_2^2 = 2 < 3$, and

$$(\varphi_{03}) \longrightarrow (\varphi_{02}, \varphi_{23}) \longrightarrow (\varphi_{01}, \varphi_{12}, \varphi_{23})$$

with $0 < \tau_2^2 = 1 < \tau_1^2 = 2 < 3$. These can generally be presented by only one maximal path as

$$(\varphi_{03}) \longrightarrow (\varphi_{0\tau_1^1}, \varphi_{\tau_1^1 3}) \longrightarrow (\varphi_{0\tau_1^2}, \varphi_{\tau_1^2\tau_2^2}, \varphi_{\tau_2^2 3})$$

where the notation "τ_j^i" considers both cases for $0 < \tau_1^2 = 1 < \tau_2^2 = 2 < 3$ and $0 < \tau_2^2 = 1 < \tau_1^2 = 2 < 3$. This argument will holds for the higher dimensional cases.

The last six 3-simplexes within $\mathbb{S}(\mathbf{Cov}_{\leq}(\mathbf{X}))(\mathcal{U}_0, \mathcal{U}_4)$ will presented by six maximal paths (refers to pages 55 and 56) with $0 < \tau_1^3, \tau_2^3, \tau_3^3 < 4$ described by

	τ_1^3	τ_2^3	τ_3^3
(1)	1	2	3
(2)	1	3	2
(3)	3	1	2
(4)	3	2	1
(5)	2	1	3
(6)	2	3	1

Furthermore, extended from the increasing structure of $0 < \tau_1^3 = 1 < \tau_2^3 = 2 < \tau_3^3 = 3 < 4$ in (1), we will have the obvious natural generalization case where $0 < \tau_1^3 = 1 < \tau_2^3 = 2 < ... < \tau_{m-1}^{m-1} = m-1 < m$, presented by the associated maximal path

$$(\varphi_{0m}) \longrightarrow (\varphi_{01}, \varphi_{12}) \longrightarrow (\varphi_{01}, \varphi_{12}, \varphi_{23}) \longrightarrow ...... \longrightarrow (\varphi_{01}, \varphi_{12}, ..., \varphi_{m-1m}).$$

So, the desired structure of this general carrier, that is $(m-1)$-simplex $\widetilde{\sigma}$, is given by the associated maximal path

$$(\varphi_{0m}) \longrightarrow (\varphi_{0\tau_1^1}, \varphi_{\tau_1^1 m}) \longrightarrow (\varphi_{0\tau_1^2}, \varphi_{\tau_1^2\tau_2^2}, \varphi_{\tau_2^2 m}) \longrightarrow$$
$$(\varphi_{0\tau_1^3}, \varphi_{\tau_1^3\tau_2^3}, \varphi_{\tau_2^3\tau_3^3}, \varphi_{\tau_3^3 m}) \longrightarrow ...... \longrightarrow (\varphi_{0\tau_1^{m-1}}, \varphi_{\tau_1^{m-1}\tau_2^{m-1}}, ..., \varphi_{\tau_{m-1}^{m-1} m})$$

with $0 < \tau_1^{m-1}, \tau_2^{m-1}, ..., \tau_{m-1}^{m-1} < m$.

This general carrier $\widetilde{\sigma}$ will give a $(m-1)$-order homotopy

$$C(X;\widetilde{\sigma}) : C(X;\mathcal{U}_0) \times \Delta[1]^{m-1} \longrightarrow C(X;\mathcal{U}_m)$$

such that

$$\begin{gathered} C(X;\widetilde{\sigma})(< U_0, ..., U_{i_1}, U_{i_1}, ..., U_{i_2}, U_{i_2}, ..., U_{i_3}, U_{i_3}, ..., U_n >, \\ (0, ..., 0_{i_1}, 1_{i_1}, ..., 1_{i_2}, 2_{i_2}, ..., 2_{i_2}, 3_{i_2}, ..., m-1)) = \\ < \\ \varphi_{0m}(U_0), ..., \varphi_{0m}(U_{i_1}), \varphi_{\tau_1^1 m}\varphi_{0\tau_1^1}(U_{i_1}), ..., \varphi_{\tau_1^1 m}\varphi_{0\tau_1^1}(U_{i_2}), \varphi_{\tau_2^2 m}\varphi_{\tau_1^2\tau_2^2}\varphi_{0\tau_1^2}(U_{i_2}), ..., \\ \varphi_{\tau_2^2 m}\varphi_{\tau_1^2\tau_2^2}\varphi_{0\tau_1^2}(U_{i_3}), \varphi_{\tau_3^3 m}\varphi_{\tau_3^3\tau_2^3}\varphi_{\tau_2^3\tau_1^3}\varphi_{0\tau_1^3}(U_{i_3}), \\ ......, \varphi_{\tau_{m-1}^{m-1} m} ... \varphi_{0\tau_1^{m-1}}(U_n) >. \end{gathered}$$

The minor thing to be check is the compatibility with "adjacent" simplexes (e.g for faces of σ). A face of σ corresponds to the removal of one layer of brakets. The carrier sequence of $d_i\sigma$ is still the same as that of σ except when i is maximal (i.e if σ is of dimension n, then $d_n\sigma$ will have smaller carrier as it composes within the innermost brackets and then removes them. So part of the carrier disappears depends on the bracketting). As we have already defined $C(X;-)$ consistently on all simplices with the same carrier it is only the last face that can cause problem. For instance, if $\mathcal{U}_0 \leq \mathcal{U}_1 \leq \mathcal{U}_2 \leq \mathcal{U}_3$ and also $\mathcal{U}_0 \leq \mathcal{U}_1 \leq \mathcal{U}_2' \leq \mathcal{U}_3$ are two carrier sequences, then clearly two of the simplices in the face of the corresponding squares coincide, but inspection shows that the formulae are then identical on those two faces and the same is true in general. Thus the definition is compatible with faces. For degenerate simplices, the carrier are not changed so they cause no problem.

This will gives a simplicial partial map

$$C(X;-)_{\mathcal{U}_0,\mathcal{U}_m} : \mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{U}_0,\mathcal{U}_m) \longrightarrow \mathcal{S}_{\mathcal{S}}(C(X;\mathcal{U}_0), C(X;\mathcal{U}_m)),$$

which maps all these general $(m-1)$-simplexes to the corresponding $(m-1)$-order homotopies.

The next stage of the proof must be the checking of the compatibility with composition. That is for the following diagram to be commutative

$$\begin{array}{ccc}
\mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{U},\mathcal{V}) \times \mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{V},\mathcal{W}) & \xrightarrow{comp} & \mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{U},\mathcal{W}) \\
\downarrow {\scriptstyle C(X;-)_{\mathcal{U},\mathcal{V}} \times C(X;-)_{\mathcal{V},\mathcal{W}}} & & \downarrow {\scriptstyle C(X;-)_{\mathcal{U},\mathcal{W}}} \\
\mathcal{S}_{\mathcal{S}}(C(X;\mathcal{U}),C(X;\mathcal{V})) \times \mathcal{S}_{\mathcal{S}}(C(X;\mathcal{V}),C(X;\mathcal{W})) & \xrightarrow[comp]{} & \mathcal{S}_{\mathcal{S}}(C(X;\mathcal{U}),C(X;\mathcal{W})).
\end{array}$$

Take two n-simplexes, σ in $\mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{U},\mathcal{V})$ and ρ in $\mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{V},\mathcal{W})$, such that

$$\mathcal{U}=\mathcal{U}_0 \leq \mathcal{U}_1 \leq ... \leq \mathcal{U}_{m_1}=\mathcal{V},$$
$$\mathcal{V}=\mathcal{U}_{m_1} \leq \mathcal{U}_{m_2} \leq ... \leq \mathcal{U}_{m_1+m_2}=\mathcal{W}.$$

These will gives two carriers, (m_1-1)-simplex $\widetilde{\sigma}$ and (m_2-1)-simplex $\widetilde{\rho}$, two associated maximal paths and also two higher order homotopies, presented in the following. For $\widetilde{\sigma}$, the corresponding maximal path is

$$(\varphi_{0m_1}) \longrightarrow (\varphi_{0\tau_1^1},\varphi_{\tau_1^1 m_1}) \longrightarrow (\varphi_{0\tau_1^2},\varphi_{\tau_1^2\tau_2^2},\varphi_{\tau_2^2 m_1}) \longrightarrow$$
$$(\varphi_{0\tau_1^3},\varphi_{\tau_1^3\tau_2^3},\varphi_{\tau_2^3\tau_3^3},\varphi_{\tau_3^3 m_1}) \longrightarrow ...... \longrightarrow (\varphi_{0\tau_1^{m_1-1}},\varphi_{\tau_1^{m_1-1}\tau_2^{m_1-1}},...,\varphi_{\tau_{m_1-1}^{m_1-1}m_1})$$

where $0<\tau_1^{m_1-1},\tau_2^{m_1-1},...,\tau_{m_1-1}^{m_1-1}<m_1$, and its associated (m_1-1)-order homotopy is

$$C(X;\widetilde{\sigma}): C(X;\mathcal{U}_0)\times\Delta[1]^{m_1-1} \longrightarrow C(X;\mathcal{U}_{m_1})$$

such that

$$C(X;\widetilde{\sigma})(<U_0,...,U_{i_1},U_{i_1},...,U_{i_2},U_{i_2},...,U_{i_3},U_{i_3},......,U_n>,$$
$$(0,...,0_{i_1},1_{i_1},...,1_{i_2},2_{i_2},...,2_{i_2},3_{i_2},......,m_1-1))=$$
$$<$$
$$\varphi_{0m_1}(U_0),...,\varphi_{0m_1}(U_{i_1}),\varphi_{\tau_1^1 m_1}\varphi_{0\tau_1^1}(U_{i_1}),...,\varphi_{\tau_1^1 m_1}\varphi_{0\tau_1^1}(U_{i_2}),\varphi_{\tau_2^2 m_1}\varphi_{\tau_1^2\tau_2^2}\varphi_{0\tau_1^2}(U_{i_2}),...,$$
$$\varphi_{\tau_2^2 m_1}\varphi_{\tau_1^2\tau_2^2}\varphi_{0\tau_1^2}(U_{i_3}),\varphi_{\tau_3^3 m_1}\varphi_{\tau_3^3\tau_2^3}\varphi_{\tau_2^3\tau_1^3}\varphi_{0\tau_1^3}(U_{i_3}),$$
$$......,\varphi_{\tau_{m_1-1}^{m_1-1}m_1}...\varphi_{0\tau_1^{m_1-1}}(U_n)>.$$

Similarly, For $\widetilde{\rho}$, the corresponding maximal path is

$$(\varphi_{m_1m_2}) \longrightarrow (\varphi_{m_1\eta_1^1}, \varphi_{\eta_1^1m_2}) \longrightarrow (\varphi_{m_1\eta_1^2}, \varphi_{\eta_1^2\eta_2^2}, \varphi_{\eta_2^2m_2}) \longrightarrow$$
$$(\varphi_{m_1\eta_1^3}, \varphi_{\eta_1^3\eta_2^3}, \varphi_{\eta_2^3\eta_3^3}, \varphi_{\eta_3^3m_2}) \longrightarrow ...... \longrightarrow (\varphi_{m_1\eta_1^{m_2-1}}, \varphi_{\eta_1^{m_2-1}\eta_2^{m_2-1}}, ..., \varphi_{\eta_{m_2-1}^{m_2-1}m_2})$$

where $0 < \eta_1^{m_2-1}, \eta_2^{m_2-1}, ..., \eta_{m_2-1}^{m_2-1} < m_2$, and its associated (m_1-1)-order homotopy is

$$C(X;\widetilde{\rho}) : C(X;\mathcal{U}_{m_1}) \times \Delta[1]^{m_2-1} \longrightarrow C(X;\mathcal{U}_{m_2})$$

such that

$$C(X;\widetilde{\rho})(< U_0, ..., U_{i_1}, U_{i_1}, ..., U_{i_2}, U_{i_2}, ..., U_{i_3}, U_{i_3}, ..., U_n >,$$
$$(0, ..., 0_{i_1}, 1_{i_1}, ..., 1_{i_2}, 2_{i_2}, ..., 2_{i_2}, 3_{i_2}, ..., m_1 - 1)) =$$
$$<$$
$$\varphi_{m_1m_2}(U_0), ..., \varphi_{m_1m_2}(U_{i_1}), \varphi_{\eta_1^1m_2}\varphi_{m_1\eta_1^1}(U_{i_1}), ..., \varphi_{\eta_1^1m_2}\varphi_{m_1\eta_1^1}(U_{i_2}), \varphi_{\eta_2^2m_2}\varphi_{\eta_1^2\eta_2^2}\varphi_{m_1\eta_1^2}(U_{i_2}), ...,$$
$$\varphi_{\eta_2^2m_2}\varphi_{\eta_1^2\eta_2^2}\varphi_{m_1\eta_1^2}(U_{i_3}), \varphi_{\eta_3^3m_2}\varphi_{\eta_3^3\eta_2^3}\varphi_{\eta_2^3\eta_1^3}\varphi_{m_1\eta_1^3}(U_{i_3}),$$
$$......, \varphi_{\eta_{m_2-1}^{m_2-1}m_2}...\varphi_{m_1\eta_1^{m_2-1}}(U_n) >.$$

The previous informations, i.e. concerning $\widetilde{\sigma}$ and $\widetilde{\rho}$ together with the (m_1-1)-order homotopy and the (m_2-1)-order homotopy, gives the left-hand side arrow of the diagram. A natural pairing

$$\Delta[1]^{m_1-1} \times \Delta[1]^{m_2-1} \longrightarrow \Delta[1]^{m_1+m_2-1}$$

then gives the bottom arrow of the diagram. Our next arguments will clarify the top and the right-hand side arrows.

By putting $0 < \tau_1^{m_1-1}, \tau_2^{m_1-1}, ..., \tau_{m_1-1}^{m_1-1} < m_1 < \eta_1^{m_2-1}, \eta_2^{m_2-1}, ..., \eta_{m_2-1}^{m_2-1} < m_2$ together and completely reindexing all the terms in between 0 and m_2, we will get

$$0 < \gamma_1^{m_1+m_2-1}, \gamma_2^{m_1+m_2-1}, ..., \gamma_{m_1+m_2-1}^{m_1+m_2-1} < m_1 + m_2$$

which indicates the following maximal path

$$(\varphi_{0m_1+m_2}) \longrightarrow (\varphi_{0\gamma_1^1}, \varphi_{\gamma_1^1m_1+m_2}) \longrightarrow (\varphi_{0\gamma_1^2}, \varphi_{\gamma_1^2\gamma_2^2}, \varphi_{\gamma_2^2m_1+m_2}) \longrightarrow$$
$$(\varphi_{0\gamma_1^3}, \varphi_{\gamma_1^3\gamma_2^3}, \varphi_{\gamma_2^3\gamma_3^3}, \varphi_{\gamma_3^3m_1+m_2}) \longrightarrow ...... \longrightarrow$$
$$(\varphi_{0\gamma_1^{m_1+m_2-1}}, \varphi_{\gamma_1^{m_1+m_2-1}\gamma_2^{m_1+m_2-1}}, ..., \varphi_{\gamma_{m_1+m_2-1}^{m_1+m_2-1}m_1+m_2}).$$

These defines the (top arrow) composite of $\widetilde{\sigma}$ and $\widetilde{\rho}$, denoted by $\widetilde{\sigma\rho}$.

This composite $\widetilde{\sigma\rho}$ later, under $C(X;-)_{\mathcal{U}_0,\mathcal{U}_{m_1+m_2}}$ (right-hand side arrow), will give an (m_1+m_2-1)-order homotopy

$$C(X;\widetilde{\sigma\rho}) : C(X;\mathcal{U}_0)\times\Delta[1]^{m_1+m_2-1}\longrightarrow C(X;\mathcal{U}_{m_1+m_2})$$

such that

$$\begin{gathered}
C(X;\widetilde{\sigma\rho})(< U_0, ..., U_{i_1}, U_{i_1}, ..., U_{i_2}, U_{i_2}, ..., U_{i_3}, U_{i_3}, ..., U_n >,\\
(0, ..., 0_{i_1}, 1_{i_1}, ..., 1_{i_2}, 2_{i_2}, ..., 2_{i_2}, 3_{i_2}, ..., m_1-1)) =\\
< \varphi_{0m_1+m_2}(U_0), ..., \varphi_{0m_1+m_2}(U_{i_1}), \varphi_{\gamma_1^1 m_1+m_2}\varphi_{0\gamma_1^1}(U_{i_1}), ...,\\
\varphi_{\gamma_1^1 m_1+m_2}\varphi_{0\gamma_1^1}(U_{i_2}), \varphi_{\gamma_2^2 m_1+m_2}\varphi_{\gamma_1^2\gamma_2^2}\varphi_{0\gamma_1^2}(U_{i_2}), ...,\\
\varphi_{\gamma_2^2 m_1+m_2}\varphi_{\gamma_1^2\gamma_2^2}\varphi_{0\gamma_1^2}(U_{i_3}), \varphi_{\gamma_3^3 m_1+m_2}\varphi_{\gamma_3^3\gamma_2^3}\varphi_{\gamma_2^3\gamma_1^3}\varphi_{0\gamma_1^3}(U_{i_3}),\\
......, \varphi_{\gamma_{m_1+m_2-1}^{m_1+m_2-1} m_1+m_2}...\varphi_{0\gamma_1^{m_1+m_2-1}}(U_n) >.
\end{gathered}$$

And finally, there is a simplicial map

$$C(X;-)_{\mathcal{U}_0,\mathcal{U}_0} : \mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{U}_0,\mathcal{U}_0)\longrightarrow\mathcal{S}_{\mathcal{S}}(C(X;\mathcal{U}_0), C(X;\mathcal{U}_0))$$

such that the following diagram commute

$$\begin{array}{ccc}
 & & \mathbb{S}(\mathbf{Cov_{\leq}(X)})(\mathcal{U}_0,\mathcal{U}_0)\\
 & \overset{Id_{\mathcal{U}_0}}{\nearrow} & \\
\Delta[0] & & \Big\downarrow C(X;-)_{\mathcal{U}_0,\mathcal{U}_0}\\
 & \underset{Id_{C(X;\mathcal{U}_0)}}{\searrow} & \\
 & & \mathcal{S}_{\mathcal{S}}(C(X;\mathcal{U}_0), C(X;\mathcal{U}_0)).
\end{array}$$

■

Chapter 5

Étale Coverings and Homotopy Coherent Diagrams

5.1 Introduction

The first step in analysing the generalization of results achieved in Chapter 4 is by considering a transition from open coverings to étale coverings. So here in this chapter, we want to discuss a more general result than that of the main theorem on the homotopy coherent Čech complex functor

$$C(X;-) : \mathbb{S}(\mathbf{Cov_{\leq}(X)}) \longrightarrow \mathcal{S}_{\mathcal{S}}.$$

Concerning that, we will show in Section 5.6 that there is an $\mathcal{S}$-functor called *homotopy coherent étale Čech complex functor*

$$\mathcal{E}(X;-) : \mathbb{S}(\mathbf{\acute{E}tCov_{\leq}(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}},$$

where $\mathbb{S}(\mathbf{\acute{E}tCov_{\leq}(X)})$ is the DKC $\mathcal{S}$-category on $\mathbf{\acute{E}tCov_{\leq}(X)}$, the "directed" category of étale coverings, and $\mathbf{SimpShv(X)}_{\mathcal{S}}$ is an $\mathcal{S}$-category defined on $\mathbf{SimpShv(X)}$, the category of simplicial sheaves and simplicial (sheaf) maps.

For a concrete example of the above broader theory, we will show in Section 5.7 that the main result in Chapter 4 can be rearranged as a particular *homotopy coherent "inclusion" étale Čech complex functor*

$$Inc\mathcal{E}(X;-) : \mathbb{S}(\mathbf{Inc\acute{E}tCov_{\leq}(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}},$$

where $\mathbb{S}(\mathbf{Inc\acute{E}tCov_{\leq}(X)})$ is the DKC $\mathcal{S}$-category on $\mathbf{Inc\acute{E}tCov_{\leq}(X)}$, the "directed" category of "inclusion" étale coverings with its natural order. This particular example establishes the link between results related to open coverings and results related to étale coverings in this context.

We present some background on the theory of sheaves in Section 5.2 and the theory of simplicial sheaves in Section 5.3. We further discuss in Section 5.4 the constructions of a sheaf $\mathcal{P}(E_X)$ associated to any given étale covering E_X, and an étale Čech complex $\mathcal{E}(X; E_X)$ associated to an étale covering E_X. We also give in Section 5.5 the étale version of Lemma 4.3.1, that is, the étale Čech complex functor

$$\mathcal{E}(X; -) : \mathbf{\acute{E}tCov_{\leq}(X)} \longrightarrow \mathbf{Ho(SimpShv(X))}.$$

Specialised from this is the "inclusion" étale Čech complex functor

$$Inc\mathcal{E}(X; -) : \mathbf{Inc\acute{E}tCov_{\leq}(X)} \longrightarrow \mathbf{Ho(SimpShv(X))}$$

discussed in Section 5.7.

5.2 Sheaves

The modern theory of sheaves is based on the earliest definition of sheaf given by Leray and Cartan, cf. Gray [30]. Briefly, a sheaf is essentially a system of local coefficients over a space X. Such for an example, if in the usual cohomology theories we consider a space X and a group G and define cohomology group $H_n(X, G)$, then in the theory of sheaves we consider not a single group G, but a whole collection of groups G_x, one for each point $x \in X$. We follow the definition of sheaf from Mumford [41] and Johnstone [33].

Definition 5.2.1 Let X be a space. A presheaf F of sets on X consists of
(i) for each open set $U \subset X$, a set $F(U)$;
(ii) for each pairs of open sets, $V \subset U$, a restriction map $\rho_{U,V} : F(U) \longrightarrow F(V)$ satisfying:
(a) $F(\emptyset) = \emptyset$, where $\emptyset$ is the empty set,

(b) $\rho_{U,U} = id_{F(U)} : F(U) \longrightarrow F(U)$ for all U,
(c) if $U_1 \subset U_2 \subset U_3$, then

$$\begin{array}{ccc} & F(U_2) & \\ {\scriptstyle \rho_{U_3,U_2}} \nearrow & & \searrow {\scriptstyle \rho_{U_2,U_1}} \\ F(U_3) & \xrightarrow[\rho_{U_3,U_1}]{} & F(U_1) \end{array}$$

commutes.

If there is no confusion, we will use the term presheaf on X to refer to a presheaf of sets on X.

Definition 5.2.2 Suppose F, G are presheaves on X. A presheaf map $f : F \longrightarrow G$ is a collection of maps $f(U) : F(U) \longrightarrow G(U)$, for U open in X, such that if $V \subset U$,

$$\begin{array}{ccc} F(U) & \xrightarrow{f(U)} & G(U) \\ {\scriptstyle \rho_{U,V}} \downarrow & & \downarrow {\scriptstyle \rho_{U,V}} \\ F(V) & \xrightarrow[f(V)]{} & G(V) \end{array}$$

commutes.

The category of presheaves and presheaf maps on X will be denoted by **PreShv(X)**.

Definition 5.2.3 A presheaf F is a sheaf if for every collection $\{U_i\}$ of open sets in X with $U = \bigcup U_i$, the diagram

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \bigcap U_j)$$

is exact, i.e. the map

$$\prod \rho_{U,U_i} : F(U) \longrightarrow \prod F(U_i)$$

is injective, and its image is the set on which

$$\prod \rho_{U_i,U_i \bigcap U_j} : \prod_i F(U_i) \longrightarrow \prod_{i,j} F(U_i \bigcap U_j)$$

and

$$\prod \rho_{U_j,U_i \bigcap U_j} : \prod_j F(U_j) \longrightarrow \prod_{i,j} F(U_i \bigcap U_j)$$

agree.

That is, for any collection of elements $x_i \in F(U_i)$ such that $\rho_{U_i,U_i \bigcap U_j}(x_i) = \rho_{U_j,U_i \bigcap U_j}(x_j)$, for all i and j, there is a unique $x \in F(U)$ such that $\rho_{U,U_i}(x) = x_i$, for all i.

A presheaf map $f : F \longrightarrow G$ is a sheaf map if the presheaves F, G on X are sheaves on X. Denote then the category of sheaves and sheaf maps on X by **Shv(X)**. Of course **Shv(X)** is thus a full subcategory of **PreShv(X)** and the inclusion has a left adjoint.

5.3 Simplicial Sheaves

We discuss here some concepts related to the category of simplicial sheaves on X, cf. K.Brown [8] and K.Brown and Gersten [9].

5.3.1 Category of Simplicial Sheaves

Definition 5.3.1 A simplicial sheaf is a functor $K : \Delta^{op} \longrightarrow$ **Shv(X)**, and a simplicial map $f : K \longrightarrow L$ of simplicial sheaves is a natural transformation.

The resulting category of simplicial sheaves will be denoted by **Simp-Shv(X)**.

K.Brown [8] showed that one example of a category of fibrant objects is the full subcategory $\mathbf{SimpShv(X)}_f$ of **SimpShv(X)** consisting of those simplicial sheaves which stalkwise satisfy Kan's extension condition, with a map to be a fibration (or weak equivalence) if it is stalkwise a fibration (or weak equivalence) in the sense of Kan. He pointed out further that a hypercovering of X is an object K of $\mathbf{SimpShv(X)}_f$ such that the simplicial map $K \longrightarrow e$, where e is a final object, is a weak equivalence. This interesting discovery initiated our study on the theory of hypercoverings of X, discussed in Chapter 6, but this idea of his fails to give the accurate picture of the general theory of a hypercovering of a topos $\mathbf{Shv(C)}$, where $\mathbf{C}$ is a site. For that reason alone, we refer to other sources for the theory of hypercovering.

5.3.2 The $\mathcal{S}$-Category $\mathbf{SimpShv(X)}_{\mathcal{S}}$

Continuing from the discussions on $\mathcal{S}$-categories in Subsections 2.4.1-2.4.4, here we give the structure of **SimpShv(X)** as an $\mathcal{S}$-category.

Definition 5.3.2 Suppose K, L are simplicial sheaves. Then define **Simp-Shv(X)**$_{\mathcal{S}}(K, L)$ by

$$\mathbf{SimpShv(X)}_{\mathcal{S}}(K, L)_n = \mathbf{SimpShv(X)}(K \times \Delta[n], L),$$

where $\Delta[n]$ is a standard n-simplex and $(K \times \Delta[n])(U) = K(U) \times \Delta[n]$, for U an open set in X. For $f \in \mathbf{SimpShv(X)}(K \times \Delta[n], L)$ and $g \in \mathbf{SimpShv(X)}(L \times \Delta[n], M)$, the composite map gf is represented by

$$K \times \Delta[n] \xrightarrow{id \times diag} K \times \Delta[n] \times \Delta[n] \xrightarrow{f \times id} L \times \Delta[n] \xrightarrow{g} M.$$

It is then standard to show:

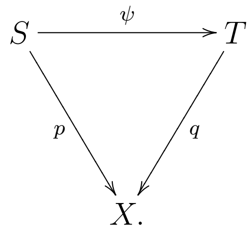
Theorem 5.3.3 $\mathbf{SimpShv(X)}_{\mathcal{S}}$ *is an* $\mathcal{S}$*-category.*

Note that **Ho(SimpShv(X))** will denote the homotopy category of simplicial sheaves and homotopy classes of simplicial sheaf maps, obtained from the obvious notion of homotopy of maps using the 1−-simplices above.

5.4 Étale Coverings and Étale Čech Complexes

The idea of an étale space and its corresponding sheaf derived from a pair of adjoint functors $\mathbf{Sets}^{\mathbf{O(X)}^{op}} \rightleftharpoons \mathbf{Top}_X$, where $\mathbf{O(X)}$ is the category of open sets of X and inclusion between them. These adjoint functors restrict to an equivalence of categories between $\mathbf{Shv(X)}$ and a full subcategory $\mathbf{\acute{E}tSp(X)}$ of $\mathbf{Top}_X$, whose objects are étale spaces and whose morphisms are étale maps, cf. Johnstone [33] and Swan [52].

Definition 5.4.1 An étale space $p : S \longrightarrow X$, denoted by S_X, is a local homeomorphism, i.e. any $x \in S$ has a neigbourhood U such that $p|_U$ maps U homeomorphically onto a neighbourhood of $p(x)$. An étale map $\psi : S_X \longrightarrow T_X$ is a commutative diagram of étale spaces



Definition 5.4.2 For any given étale space S_X, a presheaf on X is defined by

$$\mathcal{P}(S_X)(U) = \{s : U \longrightarrow S \text{ continuous}|\ ps : U \longrightarrow S \longrightarrow X \\ \text{is inclusion, i.e. such that } ps(x) = x \text{ for all } x \in U\},$$

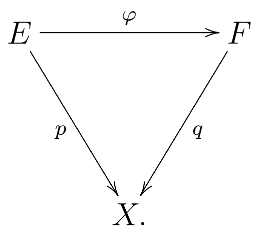
with $\rho_{UV}(s) = s|_V$, which then is a sheaf on X, the sheaf of local sections of S_X.

An example is the constant sheaf $\mathcal{P}(X)$ which is derived from a trivial étale space $id : X \longrightarrow X$.

5.4.1 Étale Coverings and Sheaf Theory

One of the applications of the above étale space is a notion of an étale covering, cf. Sullivan [50], [51], Milne [40] and Freitag and Kiehl [27]. We gives its definition below and follow it by the construction of a sheaf on X for a given étale covering.

Definition 5.4.3 An étale covering E_X is an étale space $p : E \longrightarrow X$ such that $p(E) = X$. A point in E_X, or a point in the domain E of $p : E \longrightarrow X$, is $e \in E$ such that $p(e) \in X$. An étale refinement map $\varphi_X : E_X \longrightarrow F_X$ is a commutative diagram of étale coverings



Denote then by $E_X \leq F_X$ if there is an étale refinement map $\varphi_X : E_X \longrightarrow F_X$.

The resulting ordered set (ÉtCov(X),$\leq$) can be regarded as a "directed" category denoted by **ÉtCov$_{\leq}$(X)** with ÉtCov(X) as the set of objects and with just a single morphism $E_X \leq F_X$ whenever there is an étale refinement map $\varphi_X : E_X \longrightarrow F_X$.

Examples of étale covering are numerous, but our most important one is the example that establishes the link between open coverings and étale coverings. This will be examined in detail later in Section 5.7. That is, the "inclusion" étale covering $\mathcal{U}_X$, of family of inclusion maps $\phi_i : U_i \hookrightarrow X$, represented by

$$p = \{\bigsqcup \phi_i\} : \mathcal{U} = \{\bigsqcup U_i\} \longrightarrow X$$

such that $\bigcup \phi_i(U_i) = X$. The "inclusion" étale refinement map $\psi_X : \mathcal{U}_X \longrightarrow \mathcal{V}_X$ is given by a commutative diagram

$$\begin{array}{ccc} \mathcal{U} = \{\bigsqcup U_i\} & \xrightarrow{\psi} & \mathcal{V} = \{\bigsqcup V_j\} \\ & {\scriptstyle\phi}\searrow \qquad \swarrow{\scriptstyle\theta} & \\ & X. & \end{array}$$

Analogous to a sheaf defined on an étale space, we have the corresponding definition, adapted from Definition 5.4.2.

Definition 5.4.4 For any given étale covering E_X, we get a sheaf $\mathcal{P}(E_X)$ on X, the sheaf of local sections of E_X, defined by

$$\mathcal{P}(E_X)(U) = \{\eta_U : U \longrightarrow E \text{ continuous} \mid p\eta_U : U \longrightarrow E \longrightarrow X$$
$$\text{is inclusion, i.e. such that } p\eta_U(x) = x \text{ for all } x \in U\},$$

with $\rho_{UV}(\eta_U) = \eta_V$, where $V \subset U$ and $\eta_V : V \longrightarrow E$.

One such example is the constant sheaf $\mathcal{P}(1_X)$, which is the final object of **Shv(X)**, given by the trivial étale covering $1_X : X \longrightarrow X$. It is worth also to note that a local point in $\mathcal{P}(E_X)$ is a local section $\eta_U : U \longrightarrow E$, where U is an open set in X.

Definition 5.4.5 Suppose $E_X \leq F_X$, defined by an étale refinement map $\varphi_X : E_X \longrightarrow F_X$. A sheaf map

$$\mathcal{P}(\varphi_X) : \mathcal{P}(E_X) \longrightarrow \mathcal{P}(F_X)$$

is given by $\mathcal{P}(\varphi_X)(\eta_U) = \varphi\eta_U$.

In particular, there is an (epimorphic) sheaf map

$$\mathcal{P}(p) : \mathcal{P}(E_X) \twoheadrightarrow \mathcal{P}(1_X)$$

such that $\mathcal{P}(p)(\eta_U) = p\eta_U$.

5.4.2 Étale Čech Complexes and Simplicial Sheaf Theory

Here, our motivation comes from the facts that Definition 4.2.3 gives a Čech complex $C(X;\mathcal{U})$ for an open covering $\mathcal{U}$, and Definition 5.4.4 gives a sheaf $\mathcal{P}(E_X)$, together with an epimorphism sheaf map $\mathcal{P}(p) : \mathcal{P}(E_X) \twoheadrightarrow \mathcal{P}(1_X)$, for an étale covering E_X. So, it is interesting to ask, what type of sheaf-like structure is parallel to the idea of a Čech complex $C(X;\mathcal{U})$? We examine this question below, starting with the lowest dimension, cf. Sullivan [50], [51], and Lubkin [36], [37].

For a given étale covering E_X, there is a product $E_X \times_X E_X$ with

$$E_X \times_X E_X \underset{d_0}{\overset{d_1}{\rightrightarrows}} E_X \longrightarrow 1_X .$$

The latter is associated to a product sheaf $\mathcal{P}(E_X) \times \mathcal{P}(E_X)$ with a local point looking like $< \eta_0, \eta_1 >$ over an open set U, where $p\eta_0 = p\eta_1$. This gives a 1-truncated simplicial sheaf presented as

$$\begin{array}{ccc} \mathcal{P}(E_X) \times \mathcal{P}(E_X) & \underset{d_0}{\overset{d_1}{\rightrightarrows}} & \mathcal{P}(E_X) \\ & \searrow & \downarrow \\ & & \mathcal{P}(1_X) \end{array}$$

with the associated degeneracy map given by

$$\begin{aligned} s_0 : \mathcal{P}(E_X) &\longrightarrow \mathcal{P}(E_X) \times \mathcal{P}(E_X) \\ \eta_0 &\mapsto (\eta_0, \eta_0) \end{aligned}$$

(Note, in the next more general cases, we will not include all of the degeneracy maps for simplicity of drawing).

For a given étale covering E_X, then there is a triple product $E_X \times_X E_X \times_X E_X$ with

$$E_X \times_X E_X \times_X E_X \underset{d_0}{\overset{d_2}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}}} E_X \times_X E_X \underset{d_0}{\overset{d_1}{\rightrightarrows}} E_X \longrightarrow 1_X.$$

The latter corresponds to a triple product sheaf $\mathcal{P}(E_X) \times \mathcal{P}(E_X) \times \mathcal{P}(E_X)$, with a local point presented by $< \eta_0, \eta_1, \eta_2 >$ over an open set U, where $p\eta_0 = p\eta_1 = p\eta_2$. This gives a 2-truncated simplicial sheaf presented by

$$\begin{array}{ccccc} \mathcal{P}(E_X) \times \mathcal{P}(E_X) \times \mathcal{P}(E_X) & \underset{d_0}{\overset{d_2}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}}} & \mathcal{P}(E_X) \times \mathcal{P}(E_X) & \underset{d_0}{\overset{d_1}{\rightrightarrows}} & \mathcal{P}(E_X) \\ & \searrow & & \searrow & \downarrow \\ & & & & \mathcal{P}(1_X). \end{array}$$

In general, for a given étale covering E_X, there is a n-fold product $E_X \times_X E_X \times_X ... \times E_X$ with

$$E_X \times_X ... \times_X E_X \underset{d_0}{\overset{d_n}{\substack{\longrightarrow\\ \vdots\\ \longrightarrow}}} \cdots \underset{d_0}{\overset{d_2}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}}} E_X \times_X E_X \underset{d_0}{\overset{d_1}{\rightrightarrows}} E_X \longrightarrow 1_X.$$

This will give a n-fold product sheaf $\mathcal{P}(E_X) \times \mathcal{P}(E_X) \times ... \times \mathcal{P}(E_X)$ with a local point presented by $< \eta_0, \eta_1, ..., \eta_n >$ over an open set U, where $p\eta_0 = p\eta_1 = ... = p\eta_n$. Thus, we have a n-truncated simplicial sheaf presented by

$$\begin{array}{ccccccc} \mathcal{P}(E_X) \times ... \times \mathcal{P}(E_X) & \underset{d_0}{\overset{d_n}{\substack{\longrightarrow\\ \vdots\\ \longrightarrow}}} & \cdots & \underset{d_0}{\overset{d_2}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}}} & \mathcal{P}(E_X) \times \mathcal{P}(E_X) & \underset{d_0}{\overset{d_1}{\rightrightarrows}} & \mathcal{P}(E_X) \\ & \searrow & & & & \searrow & \downarrow \\ & & & & & & \mathcal{P}(1_X). \end{array}$$

This then will form our desired structure in the following definition and parallel to the structure of the Čech complex $C(X;\mathcal{U})$.

Definition 5.4.6 For any étale covering E_X, its étale Čech complex, denoted by $\mathcal{E}(X; E_X)$, is a simplicial sheaf of the form

$$
\begin{array}{ccccc}
\cdots \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \mathcal{P}(E_X)\times\mathcal{P}(E_X) & \underset{d_0}{\overset{d_1}{\rightrightarrows}} & \mathcal{P}(E_X) \\
 & & \searrow & \downarrow \\
 & & & \mathcal{P}(1_X).
\end{array}
$$

For any $E_X \leq F_X$, defined by an étale refinement map $\varphi_X : E_X \longrightarrow F_X$, a map of étale Čech complexes is a simplicial (sheaf) map

$$\mathcal{E}(X;\varphi_X) : \mathcal{E}(X;E_X) \longrightarrow \mathcal{E}(X;F_X)$$

defined by $\mathcal{E}(X;\varphi_X)(< \eta_0, \eta_1, ... >) =< \varphi\eta_0, \varphi\eta_1, ... >$.

Due to a similar reason for the Čech complex functor case, this map is dependent on the choice of φ, but it is unique up to homotopy, so give a functor

$$\mathcal{E}(X;-) : \textbf{ÉtCov}_{\leq}\textbf{(X)} \longrightarrow \textbf{Ho(SimpShv(X))}$$

being discussed in Section 5.5 below.

It should also be noted that an étale Čech complex $\mathcal{E}(X;E_X)$ associated to any étale covering E_X is of a similar form to the object $Cosk_0\mathcal{P}(E_X)$, called *a canonical hypercovering of* X. It is an example of the more general concept of hypercovering of the topos **Shv(X)**, cf. Friedlander [28], [29] and Cox [17]. The usual notation given for its category is **CanHC(X)**, which is known to be codirected, cf. Artin, Grothendieck and Verdier [2] and Friedlander [28]. This will be examined later in Chapter 6.

5.5 Étale Čech Complex Functor

Analogous to the statement in Lemma 4.3.1, the following is an obvious generalization.

Lemma 5.5.1 If $\varphi_X, \psi_X : E_X \longrightarrow F_X$ are étale refinement maps, then there is a homotopy

$$h_X : \mathcal{E}(X; E_X) \times \Delta[1] \longrightarrow \mathcal{E}(X; F_X)$$

such that $\mathcal{E}(X; \varphi_X) \simeq \mathcal{E}(X; \psi_X)$.

Proof: The easiest way of proving this lemma is by following the idea laid in the proof of Lemma 4.3.1. Suppose that $\varphi_X, \psi_X : E_X \longrightarrow F_X$ are étale refinement maps, then there is a pair of simplicial maps

$$\mathcal{E}(X; \varphi_X), \mathcal{E}(X; \psi_X) : \mathcal{E}(X; E_X) \longrightarrow \mathcal{E}(X; F_X).$$

For a given $\sigma : \Delta[n] \longrightarrow \mathcal{E}(X; E_X)$ that picks out $< \eta_0, ..., \eta_n >$ defined over U, there exist composites

$$\mathcal{E}(X; \varphi_X)\sigma, \mathcal{E}(X; \psi_X)\sigma : \Delta[n] \longrightarrow \mathcal{E}(X; F_X)$$

again over U. This helps one to define a simplicial map

$$h_\sigma : \Delta[n] \times \Delta[1] \longrightarrow \mathcal{E}(X; F_X)$$

such that

$$h_\sigma((0, ..., t, t, ..., n), (0, ..., 0_t, 1_t, ..., 1)) = < \varphi\eta_0, ..., \varphi\eta_t, \psi\eta_t, ..., \psi\eta_n >.$$

This gives a homotopy

$$h_X : \mathcal{E}(X; \varphi_X) \simeq \mathcal{E}(X; \psi_X) : \mathcal{E}(X; E_X) \times \Delta[1] \longrightarrow \mathcal{E}(X; F_X)$$

such that

$$h_X(< \eta_0, ..., \eta_t, \eta_t, ..., \eta_n >, (0, ..., 0_t, 1_t, ..., 1)) = < \varphi\eta_0, ..., \varphi\eta_t, \psi\eta_t, ..., \psi\eta_n >.$$

■

This will define an *étale Čech complex functor*

$$\mathcal{E}(X; -) : \mathbf{\acute{E}tCov_{\leq}(X)} \longrightarrow \mathbf{Ho(SimpShv(X))}.$$

5.6 Homotopy Coherent Étale Čech Complex Functors

Parallel to the main idea in Theorem 4.4.1, there is a lift for

$$\mathcal{E}(X;-) : \mathbf{\acute{E}tCov_{\leq}(X)} \longrightarrow \mathbf{Ho(SimpShv(X))}$$

to an $\mathcal{S}$-functor

$$\mathcal{E}(X;-) : \mathbb{S}(\mathbf{\acute{E}tCov_{\leq}(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}}.$$

such that the following diagram commutes

$$\begin{array}{ccc}
\mathbb{S}(\mathbf{\acute{E}tCov_{\leq}(X)}) & \xrightarrow{\mathcal{E}(X;-)} & \mathbf{SimpShv(X)}_{\mathcal{S}} \\
\Big\downarrow{\scriptstyle aug} & & \Big\downarrow{\scriptstyle \pi_0} \\
\mathbf{\acute{E}tCov_{\leq}(X)} & \xrightarrow[\mathcal{E}(X;-)]{} & \mathbf{Ho(SimpShv(X))}.
\end{array}$$

The following result gives a more generalized form of Theorem 4.4.1.

Theorem 5.6.1 Any choice of étale refinement maps φ_X determines a lift to an $\mathcal{S}$-functor

$$\mathcal{E}(X;-) : \mathbb{S}(\mathbf{\acute{E}tCov_{\leq}(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}}.$$

Proof: By following arguments in the general proof of Theorem 4.4.1, here we can give a less detail proof for the above Theorem 5.6.1.

For an n-simplex σ in $\mathbb{S}(\mathbf{\acute{E}tCov_{\leq}(X)})(E_X, F_X)$, we can find a carrier sequence

$$\bar{\sigma} = \{E_{0_X}, E_{1_X}, ..., E_{m_X}\} \text{ with } E_X = E_{0_X} \leq E_{1_X} \leq ... \leq E_{m_X} = F_X,$$

so that the original σ is a bracketting of a string of étale refinement maps $E_{0_X} \leq E_{1_X} \leq ... \leq E_{m_X}$ of étale coverings. As before, we can thus reduce to σ being given by an $(m-1)$-simplex of $\Delta[1]^{m-1}$(representing the bracketting) so we need to define

$$\mathcal{E}(X;\bar{\sigma}) : \mathcal{E}(X;E_{0_X}) \times \Delta[1]^{m-1} \longrightarrow \mathcal{E}(X;E_{m_X}).$$

A typical $(m-1)$-simplex is given by a sequence $0 < \tau_1^{m-1}, \tau_2^{m-1}, ..., \tau_{m-1}^{m-1} < m$ giving a maximal path

$$(\varphi_{0m}) \longrightarrow (\varphi_{0\tau_1^1}, \varphi_{\tau_1^1 m}) \longrightarrow (\varphi_{0\tau_1^2}, \varphi_{\tau_1^2\tau_2^2}, \varphi_{\tau_2^2 m}) \longrightarrow$$
$$(\varphi_{0\tau_1^3}, \varphi_{\tau_1^3\tau_2^3}, \varphi_{\tau_2^3\tau_3^3}, \varphi_{\tau_3^3 m}) \longrightarrow ...... \longrightarrow (\varphi_{0\tau_1^{m-1}}, \varphi_{\tau_1^{m-1}\tau_2^{m-1}}, ..., \varphi_{\tau_{m-1}^{m-1} m}).$$

Then, there associated a $(m-1)$-order homotopy

$$\mathcal{E}(X;\bar{\sigma}) : \mathcal{E}(X;E_{0_X}) \times \Delta[1]^{m-1} \longrightarrow \mathcal{E}(X;E_{m_X})$$

such that

$$\mathcal{E}(X;\bar{\sigma})(< \eta_0, ..., \eta_{i_1}, \eta_{i_1}, ..., \eta_{i_2}, \eta_{i_2}, ..., \eta_n >,$$
$$(0, ..., 0_{i_1}, 1_{i_1}, ..., 1_{i_2}, 2_{i_2}, ..., m-1)) =$$
$$< \varphi_{0m}\eta_0, ..., \varphi_{0m}\eta_{i_1}, \varphi_{\tau_1^1 m}\varphi_{0\tau_1^1}\eta_{i_1}, ..., \varphi_{\tau_1^1 m}\varphi_{0\tau_1^1}\eta_{i_2}, \varphi_{\tau_2^2 m}\varphi_{\tau_1^2\tau_2^2}\varphi_{0\tau_1^2}\eta_{i_2},$$
$$\varphi_{\tau_2^2 m}\varphi_{\tau_1^2\tau_2^2}\varphi_{0\tau_1^2}(\eta_{i_3}), \varphi_{\tau_3^3 m}\varphi_{\tau_3^3\tau_2^3}\varphi_{\tau_2^3\tau_1^3}\varphi_{0\tau_1^3}(\eta_{i_3}),$$
$$......, \varphi_{\tau_{m-1}^{m-1} m}...\varphi_{0\tau_1^{m-1}}\eta_n >.$$

■

5.7 Application on "Inclusion" Étale Coverings

The next thing that we want to discuss in this chapter is the link between the general theory of homotopy coherent étale Čech complex functors and the particular theory of homotopy coherent "inclusion" étale Čech complex functors. The latter is achieved by considering the usual open coverings as étale coverings, and particularly, by restatement back of the main result in Chapter 4, taking an étale covering $p : E \longrightarrow X$ as an "inclusion" étale covering $\phi = \{\bigsqcup \phi_i\} : \mathcal{U} = \{\bigsqcup U_i\} \longrightarrow X$ of a family of inclusion maps $\phi_i : U_i \hookrightarrow X$, for U_i open sets.

5.7.1 "Inclusion" Étale Coverings as Sheaves and "Inclusion" Étale Čech Complexes as Simplicial Sheaves

We now considers a special case from the general idea of étale coverings.

Definition 5.7.1 An "inclusion" étale covering, denoted by $\mathcal{U}_X$, is an étale covering $\phi = \{\bigsqcup \phi_i\} : \mathcal{U} = \{\bigsqcup U_i\} \longrightarrow X$, of a family of inclusion maps $\phi_i : U_i \hookrightarrow X$, where U_i are open sets, such that $\bigcup \phi_i(U_i) = X$. A point in $\mathcal{U}_X$ is $(u_i; U_i)$ such that $\phi(u_i; U_i) = \phi_i(u_i) = u_i \in X$. An "inclusion" étale refinement map $\psi_X : \mathcal{U}_X \longrightarrow \mathcal{V}_X$ is a commutative diagram of "inclusion" étale coverings

$$\begin{array}{ccc} \mathcal{U} = \{\bigsqcup U_i\} & \xrightarrow{\ \ \psi\ \ } & \mathcal{V} = \{\bigsqcup V_j\} \\ & {}_{\phi}\searrow \quad \swarrow_{\theta} & \\ & X. & \end{array}$$

Denote $\mathcal{U}_X \leq \mathcal{V}_X$ if there is an "inclusion" étale refinement map $\varphi_X : \mathcal{U}_X \longrightarrow \mathcal{V}_X$.

The resulting "directed" category is **IncÉtCov$_{\leq}$(X)**, with the ordered set IncÉtCov(X) as the set of objects and with just a single morphism $\mathcal{U}_X \leq \mathcal{V}_X$ whenever there is an "inclusion" étale refinement map $\psi_X : \mathcal{U}_X \longrightarrow \mathcal{V}_X$.

The next step to come concerns presenting any given "inclusion" étale covering $\mathcal{U}_X$ as a sheaf $\mathcal{P}(\mathcal{U}_X)$.

Definition 5.7.2 For any given "inclusion" étale covering $\mathcal{U}_X$, a sheaf $\mathcal{P}(\mathcal{U}_X)$ is defined by

$$\begin{gathered} \mathcal{P}(\mathcal{U}_X)(U_i) = \{\eta_i : U_i \longrightarrow \mathcal{U}_X \text{ continuous} \mid \phi\eta_i : U_i \longrightarrow \mathcal{U}_X \longrightarrow X \\ \text{is inclusion, i.e. such that } \phi\eta_i(u_i) = u_i \text{ for all } u_i \in U_i\}, \end{gathered}$$

with $\rho_{U_iU_j}(\eta_i) = \eta_i|_{U_j} = \eta_j$, where $U_j \subset U_i$.

A local point in $\mathcal{P}(\mathcal{U}_X)$ is a local section $\eta_i : U_i \longrightarrow \mathcal{U}_X$. For $\mathcal{U}_X \leq \mathcal{V}_X$, the sheaf map

$$\mathcal{P}(\psi_X) : \mathcal{P}(\mathcal{U}_X) \longrightarrow \mathcal{P}(\mathcal{V}_X)$$

is given by $\mathcal{P}(\psi_X)(\eta_i) = \psi\eta_i$. Particularly, an epimorphic sheaf map

$$\mathcal{P}(\phi) : \mathcal{P}(\mathcal{U}_X) \twoheadrightarrow \mathcal{P}(1_X)$$

is given by $\mathcal{P}(\phi)(\eta_i) = \phi\eta_i$.

It is analogously sufficient then to say that we will have the idea of an "inclusion" étale Čech complex presented as a simplicial sheaf. We will adapt the lowest construction of it from the general theory in Subsection 5.4.2 and then directly pass to the general n-dimensional case.

For any "inclusion" étale covering $\mathcal{U}_X$, there is a product $\mathcal{U}_X \times_X \mathcal{U}_X$. The latter corresponds to a product sheaf $\mathcal{P}(\mathcal{U}_X) \times \mathcal{P}(\mathcal{U}_X)$ with a local pointwhich looks like $< \mu_0, \mu_1 >$ such that $\phi\mu_0 = \phi\mu_1$. This implies $\phi\mu_0(u_0) = \phi\mu_1(u_1)$, for $u_i \in U_i$, $i = 0, 1$. Later, we get $\phi\mu_0(u_0; U_0) = \phi\mu_1(u_1; U_1)$ which implies $\phi_0(u_0) = \phi_1(u_1)$. So $u_0 = u_1$ since the ϕ_i are inclusions. This means that $u_0 = u_1 \in U_0 \bigcap U_1 \neq \emptyset$.

Generally, any "inclusion" étale covering $\mathcal{U}_X$ then will give a n-fold product $\mathcal{U}_X \times_X \mathcal{U}_X \times_X ... \times_X \mathcal{U}_X$. The latter then give a n-fold product sheaf $\mathcal{P}(\mathcal{U}_X) \times \mathcal{P}(\mathcal{U}_X) \times ... \times \mathcal{P}(\mathcal{U}_X)$ with a local point presented by $< \mu_0, \mu_1, ..., \mu_n >$ such that $\phi\mu_0 = \phi\mu_1 = ... = \phi\mu_n$. This implies that $u_0 = u_1 = ... = u_n \in \bigcap_0^n U_i$. This general case then will give a n-truncated simplicial sheaf

$$\begin{array}{ccccc}
\mathcal{P}(\mathcal{U}_X) \times ... \times \mathcal{P}(\mathcal{U}_X) \; \rightrightarrows \cdots & \overset{d_2}{\underset{d_0}{\Rrightarrow}} & \mathcal{P}(\mathcal{U}_X) \times \mathcal{P}(\mathcal{U}_X) & \overset{d_1}{\underset{d_0}{\rightrightarrows}} & \mathcal{P}(\mathcal{U}_X) \\
 & \searrow & & \searrow & \downarrow \\
 & & & & \mathcal{P}(1_X).
\end{array}$$

In particular, the structure of $Inc\mathcal{E}(X;\mathcal{U}_X)$ is as follows.

Definition 5.7.3 For any "inclusion" étale covering $\mathcal{U}_X$, the "inclusion" étale Čech complex is a simplicial sheaf of the form

$$\begin{array}{ccccc} \cdots \Rrightarrow & \mathcal{P}(\mathcal{U}_X) \times \mathcal{P}(\mathcal{U}_X) & \underset{d_0}{\overset{d_1}{\rightrightarrows}} & \mathcal{P}(\mathcal{U}_X) & \\ & & \searrow & \downarrow & \\ & & & \mathcal{P}(1_X). & \end{array}$$

For any $\mathcal{U}_X \leq \mathcal{V}_X$, defined by an "inclusion" étale refinement map $\psi_X : \mathcal{U}_X \longrightarrow \mathcal{V}_X$, a map of "inclusion" étale Čech complexes is a simplicial (sheaf) map

$$Inc\mathcal{E}(X;\psi_X) : Inc\mathcal{E}(X;\mathcal{U}_X) \longrightarrow Inc\mathcal{E}(X;\mathcal{V}_X)$$

defined by $Inc\mathcal{E}(X;\psi_X)(<\mu_0,\mu_1,...>) = <\psi\mu_0,\psi\mu_1,...>$.

This map again is hopelessly dependent on choices ψ, but it is unique up to homotopy so gives

$$Inc\mathcal{E}(X;-) : \mathbf{Inc\acute{E}tCov_{\leq}(X)} \longrightarrow \mathbf{Ho(SimpShv(X))}.$$

5.7.2 "Inclusion" Étale Čech Complex Functor

Lemma 4.3.1 will have an alternative "inclusion" étale version as presented in the following.

Lemma 5.7.4 If $\psi_X, \omega_X : \mathcal{U}_X \longrightarrow \mathcal{V}_X$ are "inclusion" étale refinement maps, then there is a homotopy

$$h_X : Inc\mathcal{E}(X;\mathcal{U}_X) \times \Delta[1] \longrightarrow Inc\mathcal{E}(X;\mathcal{V}_X)$$

such that $Inc\mathcal{E}(X;\psi_X) \simeq Inc\mathcal{E}(X;\omega_X)$.

Proof: Even though that this lemma is an application of Lemma 5.5.1 and its proof can be easily constructed and understood, it is still useful for us to give the associated homotopy that completed the proof. That is, a homotopy

$$h_X : Inc\mathcal{E}(X;\varphi_X) \simeq Inc\mathcal{E}(X;\psi_X) : Inc\mathcal{E}(X;\mathcal{U}_X) \times \Delta[1] \longrightarrow Inc\mathcal{E}(X;\mathcal{V}_X)$$

such that

$$h_X(<\mu_0, ..., \mu_t, \mu_t, ..., \mu_n>, (0, ..., 0_t, 1_t, ..., 1)) = \\ <\psi\mu_0, ..., \psi\mu_t, \omega\mu_t, ..., \omega\mu_n>.$$

■

This will give a functor

$$Inc\mathcal{E}(X;-) : \mathbf{Inc\acute{E}tCov_{\leq}(X)} \longrightarrow \mathbf{Ho(SimpShv(X))}$$

which will be called an *"inclusion" étale Čech complex functor.*

5.7.3 Homotopy Coherent "Inclusion" Étale Čech Complex Functors

Applying the arguments setting in the first paragraph of Section 5.6, there is a lift of the "inclusion" étale Čech complex functor to a *homotopy coherent "inclusion" étale Čech complex functor* which make the following diagram commutative

$$\begin{array}{ccc} \mathbb{S}(\mathbf{Inc\acute{E}tCov_{\leq}(X)}) & \xrightarrow{Inc\mathcal{E}(X;-)} & \mathbf{SimpShv(X)}_{\mathcal{S}} \\ \downarrow{\scriptstyle aug} & & \downarrow{\scriptstyle \pi_0} \\ \mathbf{Inc\acute{E}tCov_{\leq}(X)} & \xrightarrow[Inc\mathcal{E}(X;-)]{} & \mathbf{Ho(SimpShv(X))}. \end{array}$$

The following is an obvious application of Theorem 5.6.1 and an alternative sheaf version of Theorem 4.4.1.

Theorem 5.7.5 Any choice of "inclusion" étale refinement mapsψ_X determines a lift to an $\mathcal{S}$-functor

$$Inc\mathcal{E}(X;-) : \mathbb{S}(\mathbf{Inc\acute{E}tCov_{\leq}(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}}.$$

Chapter 6

Hypercoverings and Homotopy Coherent Diagrams

6.1 Introduction

Earlier in Subsection 5.3.1, we stated that K. Brown's [8] notion of hypercovering of a space X presented an inaccurate picture of the general theory of hypercovering of a topos **Shv(C)**, where **C** is a site. The idea of his took a *hypercovering of* X to be an object K of $\mathbf{SimpShv(X)}_f$, the category of fibrant objects, such that the simplicial map $K \longrightarrow e$ is a weak equivalence. Artin, Grothendieck and Verdier [2], and Artin and Mazur [3], took a hypercovering of **Shv(C)** to be an object U of **SimpShv(C)** such that:
(1) For e the final object in **SimpShv(C)**, the morphism $U_0 \longrightarrow e$ is an covering,
(2) The canonical morphism

$$U_{n+1} \longrightarrow (Cosk_n U)_{n+1}$$

is an covering, for $n > 0$.

Such an example is, for an object V of **Shv(C)** with $U \twoheadrightarrow V$ epimorphism, an object $Cosk_0 U$ of **SimpShv(C)** of the form

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} U \times_V U \times_V U \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} U \times_V U \substack{\longrightarrow\\\longrightarrow} U \longrightarrow V,$$

is a hypercovering of V, cf. Pataraia [42] for this $Cosk_0U$ type of construction associated to the topos **Shv(C)**. The inaccuracy is due to the fact that the corresponding weak equivalence simplicial (sheaf) map $K \longrightarrow e$ is defined through the idea of stalkwise weak equivalences, and this particular idea of "stalkwise" does not at all appear in a general Grothendieck topos.

The more reliable source comes from Friedlander [28],[29] and Cox [17] for hypercoverings of a scheme S. Suppose **Ét(S)** is a category of schemes étale over S, and **Simp(Ét(S))** is the category of simplicial objects in **Ét(S)**. A *hypercovering of* S then is an object P of **Simp(Ét(S))** such that:
(1) For e the final object in **Simp(Ét(S))**, the map $P_0 \longrightarrow e$ is surjective,
(2) The canonical morphism

$$P_{n+1} \longrightarrow (Cosk_n P)_{n+1}$$

is surjective, for $n > 0$.

Analogous to the earlier example $Cosk_0U$ in **SimpShv(C)**, for an object Q of **Ét(S)** with $P \longrightarrow Q$ surjective, an object $Cosk_0P$ of the form

$$\cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; P \times_Q P \times_Q P \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; P \times_Q P \;\substack{\longrightarrow\\ \longrightarrow}\; P \longrightarrow Q.$$

is a hypercovering of Q.

Both ideas of hypercoverings above will be applied to define a notion of a hypercovering of a topos **Shv(X)**. A hypercovering of **Shv(X)** then is an object K of **SimpShv(X)** such that:
(1) For e the final object in **SimpShv(X)**, the map $K_0 \longrightarrow e$ is a surjective,
(2) The canonical map

$$K_{n+1} \longrightarrow (Cosk_n K)_{n+1}$$

is a surjective, for $n > 0$.

Related to the above examples, our findings in Subsection 5.4.2 on an étale Čech complex $\mathcal{E}(X; E_X)$

$$\begin{array}{ccccc} \cdots \Rrightarrow & \mathcal{P}(E_X) \times \mathcal{P}(E_X) & \rightrightarrows & \mathcal{P}(E_X) \\ & & \searrow & \downarrow \\ & & & \mathcal{P}(1_X), \end{array}$$

and on an "inclusion" étale Čech complex $Inc\mathcal{E}(X;\mathcal{U}_X)$

$$\begin{array}{ccccc} \cdots \Rrightarrow & \mathcal{P}(\mathcal{U}_X) \times \mathcal{P}(\mathcal{U}_X) & \rightrightarrows & \mathcal{P}(\mathcal{U}_X) \\ & & \searrow & \downarrow \\ & & & \mathcal{P}(1_X), \end{array}$$

are of the form similar to this type of hypercovering of L. They will be called a canonical hypercovering of X and an "inclusion" canonical hypercovering of X. Compare both to the idea of a hypercovering of X by K.Brown [8]. The corresponding category **CanHCov(X)** and category **IncCanHCov(X)** are known to be codirected, repeating the same arguments in Chapter 5. Further, the associated $\mathcal{S}$-categories $\mathbf{CanHCov(X)}_{\mathcal{S}}$ and $\mathbf{IncCanHCov(X)}_{\mathcal{S}}$ are embedded in $\mathbf{SimpShv(X)}_{\mathcal{S}}$, and π_0 maps them to give cofiltering categories **Ho(CanHCov(X))** and **Ho(IncCanHCov(X))**, cf. Friedlander [28]. We conjectured that the canonical hypercovering functor

$$\mathbf{Ho(CanHCov(X))} \longrightarrow \mathbf{Ho(SimpShv(X))}$$

can be lifted to the homotopy coherent canonical hypercovering functor

$$\mathbb{S}(\mathbf{CanHCov(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}},$$

and in particular, the "inclusion" canonical hypercovering functor

$$\mathbf{Ho(IncCanHCov(X))} \longrightarrow \mathbf{Ho(SimpShv(X))}$$

can be lifted to the homotopy coherent "inclusion" canonical hypercovering functor

$$\mathbb{S}(\mathbf{IncCanHCov(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}}.$$

These will be discuss in detail in Section 6.2.

Based on the above developments, Section 6.3 will present a more general case related to the idea of hypercoverings of a topos **Shv(X)**. That is, the conjecture that the hypercovering functor

$$\mathbf{Ho(HCov(Shv(X)))} \longrightarrow \mathbf{Ho(SimpShv(X))}$$

can be lifted to the homotopy coherent hypercovering functor

$$\mathbb{S}(\mathbf{HCov(Shv(X))}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}}.$$

Later on, by considering the more general result related to hypercoverings of a topos **Shv(C)**, Section 6.4 will present a conjecture that the Grothendieck-Verdier hypercovering functor

$$\mathbf{Ho(HCov(Shv(C)))} \longrightarrow \mathbf{Ho(SimpShv(C))}$$

can be lifted to the homotopy coherent Grothendieck-Verdier hypercovering functor

$$\mathbb{S}(\mathbf{HCov(Shv(C))}) \longrightarrow \mathbf{SimpShv(C)}_{\mathcal{S}}.$$

6.2 Homotopy Coherent Canonical Hypercovering Functor

In the following, we will present an alternative definition of an étale Čech complex $\mathcal{E}(X;E_X)$ and an "inclusion" étale Čech complex $Inc\mathcal{E}(X;\mathcal{U}_X)$.

Definition 6.2.1 A canonical hypercovering of X is an étale Čech complex $\mathcal{E}(X;E_X)$ of the form

$$\begin{array}{ccccc} \cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \mathcal{P}(E_X)\times\mathcal{P}(E_X) & \rightrightarrows & \mathcal{P}(E_X) \\ & & \searrow & \downarrow \\ & & & \mathcal{P}(1_X). \end{array}$$

An “inclusion” canonical hypercovering of X is an “inclusion” étale Čech complex $Inc\mathcal{E}(X;\mathcal{U}_X)$ *of the form*

$$\begin{array}{ccccc} \cdots \rightrightarrows & \mathcal{P}(\mathcal{U}_X)\times\mathcal{P}(\mathcal{U}_X) & \rightrightarrows & \mathcal{P}(\mathcal{U}_X) \\ & & \searrow & \downarrow \\ & & & \mathcal{P}(1_X). \end{array}$$

Consequently, we will have the following obvious lemma.

Lemma 6.2.2 A canonical hypercovering $\mathcal{E}(X;E_X)$ of X is a hypercovering $Cosk_0\mathcal{P}(E_X)$ of X. An “inclusion” canonical hypercovering $Inc\mathcal{E}(\mathcal{U}_X)$ of X is a hypercovering $Cosk_0\mathcal{P}(\mathcal{U}_X)$ of X.

Proof: Considers $\mathcal{P}(1_X)$ as the final object e in $\mathbf{SimpShv(X)}_{\mathcal{S}}$. ■

From now on, we denote a canonical hypercovering of X as $Cosk_0\mathcal{P}(E_X)$, and an “inclusion” canonical hypercovering of X as $Cosk_0\mathcal{P}(\mathcal{U}_X)$. Related to that, Friedlander [28] provided us with an idea of a morphism between hypercoverings of a scheme, and we will apply this idea of his to consider a morphism between canonical hypercoverings of X.

Definition 6.2.3 For an étale refinement map $\varphi_X : E_X \longrightarrow F_X$, a morphism

$$Cosk_0\mathcal{P}(\varphi_X) : Cosk_0\mathcal{P}(E_X) \longrightarrow Cosk_0\mathcal{P}(F_X)$$

between canonical hypercoverings of X is a map of étale Čech complexes, satisfying the following conditions:
(i) The map $\mathcal{P}(E_X) \longrightarrow \mathcal{P}(F_X)$ is surjective, and
(ii) The map

$$(Cosk_0\mathcal{P}(E_X))_{n+1} \longrightarrow (Cosk_0\mathcal{P}(E_X))_{n+1} \times_{(Cosk_0\mathcal{P}(F_X))_{n+1}} (Cosk_0\mathcal{P}(F_X))_{n+1}$$

is surjective, for $n > 0$.
Similar idea of definition holds for a morphism between “inclusion” canonical hypercoverings of X.

These define a category **CanHC(X)** of canonical hypercoverings of X and its morphisms, and a category **IncCanHC(X)** of "inclusion" canonical hypercoverings of X and its morphisms, which are known and stated before as to be codirected, cf Friedlander [28].

Lemma 6.2.4 The categories **CanHCov(X)** and **IncCanHCov(X)** are codirected.

By adjusting the proofs of Lemma 5.5.1 and Lemma 5.7.4in an obvious way, we will have the following result.

Lemma 6.2.5 (i) For $\varphi_X, \psi_X : E_X \longrightarrow F_X$ étale refinement maps, there is a homotopy

$$Cosk_0\mathcal{P}(\varphi_X) \simeq Cosk_0\mathcal{P}(\psi_X) : Cosk_0\mathcal{P}(E_X) \times \Delta[1] \longrightarrow Cosk_0\mathcal{P}(F_X).$$

(ii) For $\psi_X, \omega_X : \mathcal{U}_X \longrightarrow \mathcal{V}_X$ "inclusion" étale refinement maps, there is a homotopy

$$Cosk_0\mathcal{P}(\psi_X) \simeq Cosk_0\mathcal{P}(\omega_X) : Cosk_0\mathcal{P}(\mathcal{U}_X) \times \Delta[1] \longrightarrow Cosk_0\mathcal{P}(\mathcal{V}_X).$$

Proof: Applying Lemma 5.5.1 by considering $\mathcal{E}(X; E_X)$ as $Cosk_0\mathcal{P}(E_X)$, and Lemma 5.7.4 by considering $\mathcal{E}(X;\mathcal{U}_X)$ as $Cosk_0\mathcal{P}(\mathcal{U}_X)$. ■

These define a *canonical hypercovering functor*

$$Cosk_0\mathcal{P}(-) : \mathbf{\acute{E}tCov_{\leq}(X)} \longrightarrow \mathbf{Ho(CanHCov(X))}$$

where **Ho(CanHCov(X))** is a category of canonical hypercoverings of X and its homotopy classes of morphisms, and an *"inclusion" canonical hypercovering functor*

$$Cosk_0\mathcal{P}(-) : \mathbf{Inc\acute{E}tCov_{\leq}(X)} \longrightarrow \mathbf{Ho(IncCanHCov(X))}$$

where **Ho(IncCanHCov(X))** is a category of "inclusion" canonical hypercoverings of X and its homotopy classes of morphisms.

The canonical hypercovering functor then can be lifted to a *homotopy coherent canonical hypercovering functor* such that the following diagram

$$
\begin{array}{ccc}
\mathbb{S}(\mathbf{\acute{E}tCov}_{\leq}(\mathbf{X})) & \xrightarrow{Cosk_0\mathcal{P}(-)} & \mathbf{CanHC(X)}_{\mathcal{S}} \\
\Big\downarrow{\scriptstyle aug} & & \Big\downarrow{\scriptstyle \pi_0} \\
\mathbf{\acute{E}tCov}_{\leq}(\mathbf{X}) & \xrightarrow[Cosk_0\mathcal{P}(-)]{} & \mathbf{Ho(CanHC(X))}
\end{array}
$$

commutes, and the "inclusion" canonical hypercovering functor then can be lifted to a *homotopy coherent "inclusion" canonical hypercovering functor* such that the following diagram

$$
\begin{array}{ccc}
\mathbb{S}(\mathbf{Inc\acute{E}tCov}_{\leq}(\mathbf{X})) & \xrightarrow{Cosk_0\mathcal{P}(-)} & \mathbf{IncCanHCov(X)}_{\mathcal{S}} \\
\Big\downarrow{\scriptstyle aug} & & \Big\downarrow{\scriptstyle \pi_0} \\
\mathbf{Inc\acute{E}tCov}_{\leq}(\mathbf{X}) & \xrightarrow[Cosk_0\mathcal{P}(-)]{} & \mathbf{Ho(IncCanHCov(X))}
\end{array}
$$

commute.

These gives a restatement of Theorem 5.6.1 and Theorem 5.7.5 in a hypercovering version.

Theorem 6.2.6 (i) Any choice of étale refinement maps φ_X determines a lift to an $\mathcal{S}$-functor

$$Cosk_0\mathcal{P}(-) : \mathbb{S}(\mathbf{\acute{E}tCov}_{\leq}(\mathbf{X})) \longrightarrow \mathbf{CanHCov(X)}_{\mathcal{S}}.$$

(ii) Any choice of étale refinement maps ψ_X determines a lift to an $\mathcal{S}$-functor

$$Cosk_0\mathcal{P}(-) : \mathbb{S}(\mathbf{Inc\acute{E}tCov}_{\leq}(\mathbf{X})) \longrightarrow \mathbf{IncCanHCov(X)}_{\mathcal{S}}.$$

Proof: Applying Theorem 5.6.1 by considering $\mathcal{E}(X; E_X)$ as $Cosk_0\mathcal{P}(E_X)$, and Theorem 5.7.5 by considering $\mathcal{E}(X; \mathcal{U}_X)$ as $Cosk_0\mathcal{P}(\mathcal{U}_X)$. ∎

By comparing the above results to the results in Theorem 5.6.1, i.e. on the homotopy coherent étale Čech complex functor

$$\mathcal{E}(X;-) : \mathbb{S}(\mathbf{\acute{E}tCov_{\leq}(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}},$$

and in Theorem 5.7.2, i.e. on the homotopy coherent "inclusion" étale Čech complex functor

$$Inc\mathcal{E}(X;-) : \mathbb{S}(\mathbf{Inc\acute{E}tCov_{\leq}(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}},$$

it is seen that the $\mathcal{S}$-categories $\mathbf{CanHCov(X)}_{\mathcal{S}}$ and $\mathbf{IncCanHCov(X)}_{\mathcal{S}}$ are embedded in $\mathbf{SimpShv(X)}_{\mathcal{S}}$. These $\mathcal{S}$-categories are mapped then by π_0 to give $\mathbf{Ho(CanHCov(X))}$, $\mathbf{Ho(IncCanHCov(X))}$ and $\mathbf{Ho(SimpShv(X))}$. So, we will have the following (combined) diagram

$$\begin{array}{ccccc}
\mathbf{CanHCov(X)}_{\mathcal{S}} & \hookrightarrow & \mathbf{SimpShv(X)}_{\mathcal{S}} & \hookleftarrow & \mathbf{IncCanHCov(X)}_{\mathcal{S}} \\
\downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\
\mathbf{Ho(CanHCov(X))} & \longrightarrow & \mathbf{Ho(SimpShv(X))} & \longleftarrow & \mathbf{Ho(IncCanHCov(X))}.
\end{array}$$

With the similar type of argument on lifting those "bottom" functors to $\mathcal{S}$-functors, we will have then the following commutative diagram

$$\begin{array}{ccccc}
\mathbb{S}(\mathbf{CanHCov(X)}) & \longrightarrow & \mathbf{SimpShv(X)}_{\mathcal{S}} & \longleftarrow & \mathbb{S}(\mathbf{IncCanHCov(X)}) \\
\downarrow{\scriptstyle aug} & & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle aug} \\
\mathbf{Ho(CanHCov(X))} & \longrightarrow & \mathbf{Ho(SimpShv(X))} & \longleftarrow & \mathbf{Ho(IncCanHCov(X))}.
\end{array}$$

This resulted in giving us the following conjectures.

Conjecture 6.2.7 (i) The functor

$$\mathbf{Ho(CanHCov(X))} \longrightarrow \mathbf{Ho(SimpShv(X))}$$

can be lifted to an $\mathcal{S}$-functor

$$\mathbb{S}(\mathbf{CanHCov(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}}.$$

(ii) The functor

$$\mathbf{Ho(IncCanHCov(X))} \longrightarrow \mathbf{Ho(SimpShv(X))}$$

can be lifted to an $\mathcal{S}$-functor

$$\mathbb{S}(\mathbf{IncCanHCov(X)}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}}.$$

6.3 Homotopy Coherent Hypercovering Functor

The facts in Section 6.2 shows that an étale Čech complex $\mathcal{E}(X; E_X)$ is a canonical hypercovering $Cosk_0\mathcal{P}(E_X)$ of X, and an "inclusion" étale Čech complex $\mathbf{Inc}\mathcal{E}(X;\mathcal{U}_X)$ is an "inclusion" canonical hypercovering $Cosk_0\mathcal{P}(\mathcal{U}_X)$ of X. Further facts revealed that both are of the form similar to an example of a hypercovering of $\mathbf{Shv(X)}$, i.e. for an object L in $\mathbf{Shv(X)}$ with $K \longrightarrow L$ surjective, $Cosk_0K$ is a hypercovering of L. Motivated by these, here we present the more general idea of a hypercovering of the topos $\mathbf{Shv(X)}$ and later develop the similar ideas on it.

Definition 6.3.1 An object K of $\mathbf{SimpShv(X)}$ is called a hypercovering of $\mathbf{Shv(X)}$ if:
(i) For e the final object in $\mathbf{SimpShv(X)}$, the map $K_0 \longrightarrow e$ is surjective.
(ii) The canonical map

$$K_{n+1} \longrightarrow (Cosk_nK)_{n+1}$$

is surjective, for $n > 0$.

The morphism between hypercoverings of $\mathbf{Shv(X)}$ can easily be generalized from Definition 6.2.3, cf. Friedlander [28].

Definition 6.3.2 A morphism of hypercoverings K and L of $\mathbf{Shv(X)}$ is a surjective simplicial map $f : K \longrightarrow L$ such that
(i) The map $K_0 \longrightarrow L_0$ is surjective.
(ii) The map

$$K_{n+1} \longrightarrow (Cosk_nK)_{n+1} \times_{(Cosk_nL)_{n+1}} L_{n+1}$$

is surjective, for $n > 0$.

The associated category of hypercoverings of $\mathbf{Shv(X)}$ and its morphisms will be denoted by $\mathbf{HCov(Shv(X))}$, which is known to be codirected, cf. Friedlander [28].

Lemma 6.3.3 The category $\mathbf{HCov(Shv(X))}$ is codirected.

Using the similar construction for forming an $\mathcal{S}$-category as in Section 2.4, it is revealed that $\mathbf{HCov(Shv(X))}_{\mathcal{S}}$ is embedded in $\mathbf{SimpShv(X)}_{\mathcal{S}}$, and the functor π_0 maps it to $\mathbf{Ho(HCov(Shv(X)))}$, the cofiltering category of hypercoverings of $\mathbf{Shv(X)}$ and homotopy classes of morphisms. Together with the information that π_0 also maps $\mathbf{SimpShv(X)}_{\mathcal{S}}$ to $\mathbf{Ho(SimpShv(X))}$, we will have the following diagram

$$\begin{array}{ccc} \mathbf{HCov(Shv(X))}_{\mathcal{S}} & \hookrightarrow & \mathbf{SimpShv(X)}_{\mathcal{S}} \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\ \mathbf{Ho(HC(Shv(X)))} & \longrightarrow & \mathbf{Ho(SimpShv(X))}. \end{array}$$

We conjectured that the "bottom" part of the diagram will be lifted to an $\mathcal{S}$-functor called *homotopy coherent hypercovering functor* which induced a commutative diagram below

$$\begin{array}{ccc} \mathbb{S}(\mathbf{HCov(Shv(X))}) & \longrightarrow & \mathbf{SimpShv(X)}_{\mathcal{S}} \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\ \mathbf{Ho(HCov(Shv(X)))} & \longrightarrow & \mathbf{Ho(SimpShv(X))}. \end{array}$$

This can be simplified in the following statement.

Conjecture 6.3.4 The functor

$$\mathbf{Ho(HCov(Shv(X)))} \longrightarrow \mathbf{Ho(SimpShv(X))}$$

can be lifted to an $\mathcal{S}$-functor

$$\mathbb{S}(\mathbf{HCov(Shv(X))}) \longrightarrow \mathbf{SimpShv(X)}_{\mathcal{S}}.$$

6.4 Homotopy Coherent Grothendieck-Verdier Hypercovering Functor

Our purpose here is to argue in a similar fashion as in Section 6.3, but in broader context, i.e. considering the idea of hypercovering of the topos **Shv(C)**, where **C** is a site, instead of the topos **Shv(X)**. Preliminaries to that are all the concepts build on a site **C**, cf. Artin, Grothendieck and Verdier [2] and Johnstone [33].

6.4.1 Grothendieck Topos

Let **C** be a small category with pullbacks.

Definition 6.4.1 A Grothendieck pretopology on **C** is defined by specifying, for each object U of **C**, a set $P(U)$ of families of morphisms of the form $\{\alpha_i : U_i \longrightarrow U | i \in I\}$, called covering families of the pretopology, such that
(i) For any U, the family whose only member $1 : U \longrightarrow U$ is in $P(U)$.
(ii) If $V \longrightarrow U$ is a morphism in **C** and $\{U_i \longrightarrow U | i \in I\}$ is in $P(U)$, then the family of morphisms $\{\pi_i : V \times U_i \longrightarrow V | i \in I\}$ is in $P(V)$.
(iii) If $\{\alpha_i : U_i \longrightarrow U | i \in I\} \in P(U)$ and $\{\beta_{ij} : V_{ij} \longrightarrow U_i | j \in J_i\} \in P(U_i)$ for each i, then $\{\alpha_i\beta_{ij} : V_{ij} \longrightarrow U | i \in I, j \in J_i\} \in P(U)$.

We could define a sheaf for the pretopology P, but there is a certain imprecision in the definition of a Grothendieck pretopology, in that two different pretopologies may give exactly the same sheaves. To remove this, we restrict our attention to those families, say R, which are "saturated" in the sense that $\{\alpha : V \longrightarrow U\} \in R$ implies $\{\alpha\beta : W \longrightarrow U\} \in R$ for any $\beta : W \longrightarrow V$. Such a family is called a *sieve* on the object U.

Definition 6.4.2 A Grothendieck topology on **C** is defined by specifying, for each object U of **C**, a set $J(U)$ of sieves on U, called covering sieves of the topology, such that
(i) For any U, the maximal sieve $\{\alpha | cod(\alpha) = U\}$ is in $J(U)$.
(ii) If $R \in J(U)$ and $f : V \longrightarrow U$ is a morphism of **C**, then the induces sieve given by $f^*(R) = \{\alpha : W \longrightarrow V | f\alpha \in R\}$ is in $J(V)$.

(iii) If $R \in J(U)$ and S is a sieve on U such that, for each $f : V \longrightarrow U$ in R, we have $f^*(S) \in J(V)$, then $S \in J(U)$.
A small category equipped with a Grothendieck topology is called a site.

The "sheaf-like" property on a site which is parallel to the idea of a sheaf on a topological space will give us the notion of Grothendieck topos. Suppose **C** is a site and F is a presheaf on **C** such that the diagram

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is an equalizer for every covering family $\{U_i \longrightarrow U | i \in I\}$.

Definition 6.4.3 A presheaf F is a sheaf if, for every object U of **C** and every $R \in J(U)$, each morphism $R \longrightarrow F$ in $\mathbf{Sets}^{\mathbf{C}^{op}}$ has exactly one extension to a morphism $h_U \longrightarrow F$.

We denote the full subcategory of $\mathbf{Sets}^{\mathbf{C}^{op}}$ whose objects are J-sheaves by **Shv(C)**.

Definition 6.4.4 A Grothendieck topos is a category of sheaves on **C**, denoted by **Shv(C)**.

6.4.2 The $\mathcal{S}$-Category $\mathbf{SimpShv(C)}_{\mathcal{S}}$

The discussions in K.Brown [8], K.Brown and Gersten [9] and in Subsection 5.3.2 on the concept of simplicial sheaves on X can be generalized to the notion of simplicial sheaves on a site **C**, cf. Jardine [31], [32], and Crans [18]. Suppose **Shv(C)** is a Grothendieck topos, i.e. a category of sheaves on **C**.

Definition 6.4.5 A simplicial sheaf on **C** is a functor $K : \Delta^{op} \longrightarrow \mathbf{Shv(C)}$, and a simplicial map $f : K \longrightarrow L$ of simplicial sheaves is a natural transformation.

The resulting category will be denoted by **SimpShv(C)**. Generally, Definition 5.3.2 and Theorem 5.3.3 can be extends to the following statements.

Definition 6.4.6 Suppose K, L are simplicial sheaves. Define $\mathbf{SimpShv(C)}_{\mathcal{S}}(K, L)$ by

$$\mathbf{SimpShv(C)}_{\mathcal{S}}(K, L)_n = \mathbf{SimpShv(C)}(K \times \Delta[n], L),$$

where $\Delta[n]$ is a standard n-simplex, and $(K \times \Delta[n])(U) = K(U) \times \Delta[n]$, for U is an object of $\mathbf{C}$. For $f \in \mathbf{SimpShv(C)}(K \times \Delta[n], L)$ and $g \in \mathbf{SimpShv(C)}(L \times \Delta[n], M)$, the composite map gf is presented by

$$K \times \Delta[n] \xrightarrow{id \times diag} K \times \Delta[n] \times \Delta[n] \xrightarrow{f \times id} L \times \Delta[n] \xrightarrow{g} M.$$

The following is a standard theorem.

Theorem 6.4.7 $\mathbf{SimpShv(C)}_{\mathcal{S}}$ is an $\mathcal{S}$-category.

6.4.3 Main Conjecture

We give first all the necessary materials needed for the development of our general result on the homotopy coherent Grothendieck-Verdier hypercovering functor, cf. Artin, Grothendieck and Verdier [2] and Artin and Mazur [3]. Suppose **Shv(C)** is a Grothendieck topos and **SimpShv(C)** is the corresponding category of simplicial sheaves.

Definition 6.4.8 An object X of **SimpShv(C)** is called a hypercovering of **Shv(C)** if:
(i) For e the final object in **SimpShv(C)**, the morphism $X_0 \longrightarrow e$ is a covering.
(ii) The canonical morphism

$$X_{n+1} \longrightarrow (Cosk_n X)_{n+1}$$

is a covering, for $n > 0$.

The morphism between hypercoverings of **Shv(C)** then is a simplicial map which satisfying some properties as before.

Definition 6.4.9 A morphism of hypercoverings X and Y of **Shv(C)** is a simplicial map $f : X \longrightarrow Y$ such that:
(i) The morphism $X_0 \longrightarrow Y_0$ is a covering.
(ii) The morphism

$$X_{n+1} \longrightarrow (Cosk_n X)_{n+1} \times_{(Cosk_n Y)_{n+1}} Y_{n+1}$$

is a covering, for $n > 0$.

The associated category will be denoted by **HCov(Shv(C))**, and it is easily extended from Lemma 6.3.3 that it is codirected, cf. Artin, Grothendieck and Verdier [2]. Analogously, the $\mathcal{S}$-category $\mathbf{HCov(Shv(C))}_{\mathcal{S}}$ is embedded in $\mathbf{SimpShv(C)}_{\mathcal{S}}$, and the π_0 functor mapped it to a cofiltering category **Ho(HCov(Shv(C)))**, presented in the following diagram

$$\begin{array}{ccc} \mathbf{HCov(Shv(C))}_{\mathcal{S}} & \hookrightarrow & \mathbf{SimpShv(C)}_{\mathcal{S}} \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0} \\ \mathbf{Ho(HC(Shv(C)))} & \longrightarrow & \mathbf{Ho(SimpShv(C))}. \end{array}$$

We cojectured that the "bottom" part of the above diagram will be lifted to an $\mathcal{S}$-functor called *homotopy coherent Grothendieck-Verdier hypercovering functor* which induces the following commutative diagram

$$\begin{array}{ccc} \mathbb{S}(\mathbf{HCov(Shv(C))}) & \longrightarrow & \mathbf{SimpShv(C)}_{\mathcal{S}} \\ \downarrow{\scriptstyle aug} & & \downarrow{\scriptstyle \pi_0} \\ \mathbf{Ho(HCov(Shv(C)))} & \longrightarrow & \mathbf{Ho(SimpShv(C))}. \end{array}$$

The following is a summary of this conjecture.

Conjecture 6.4.10 The functor

$$\mathbf{Ho(HCov(Shv(C)))} \longrightarrow \mathbf{Ho(SimpShv(C))}$$

can be lifted to an $\mathcal{S}$-functor

$$\mathbb{S}(\mathbf{HCov(Shv(C))}) \longrightarrow \mathbf{SimpShv(C)}_{\mathcal{S}}.$$

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